Resurgence, TQFT and fractional instantons in gauge theory

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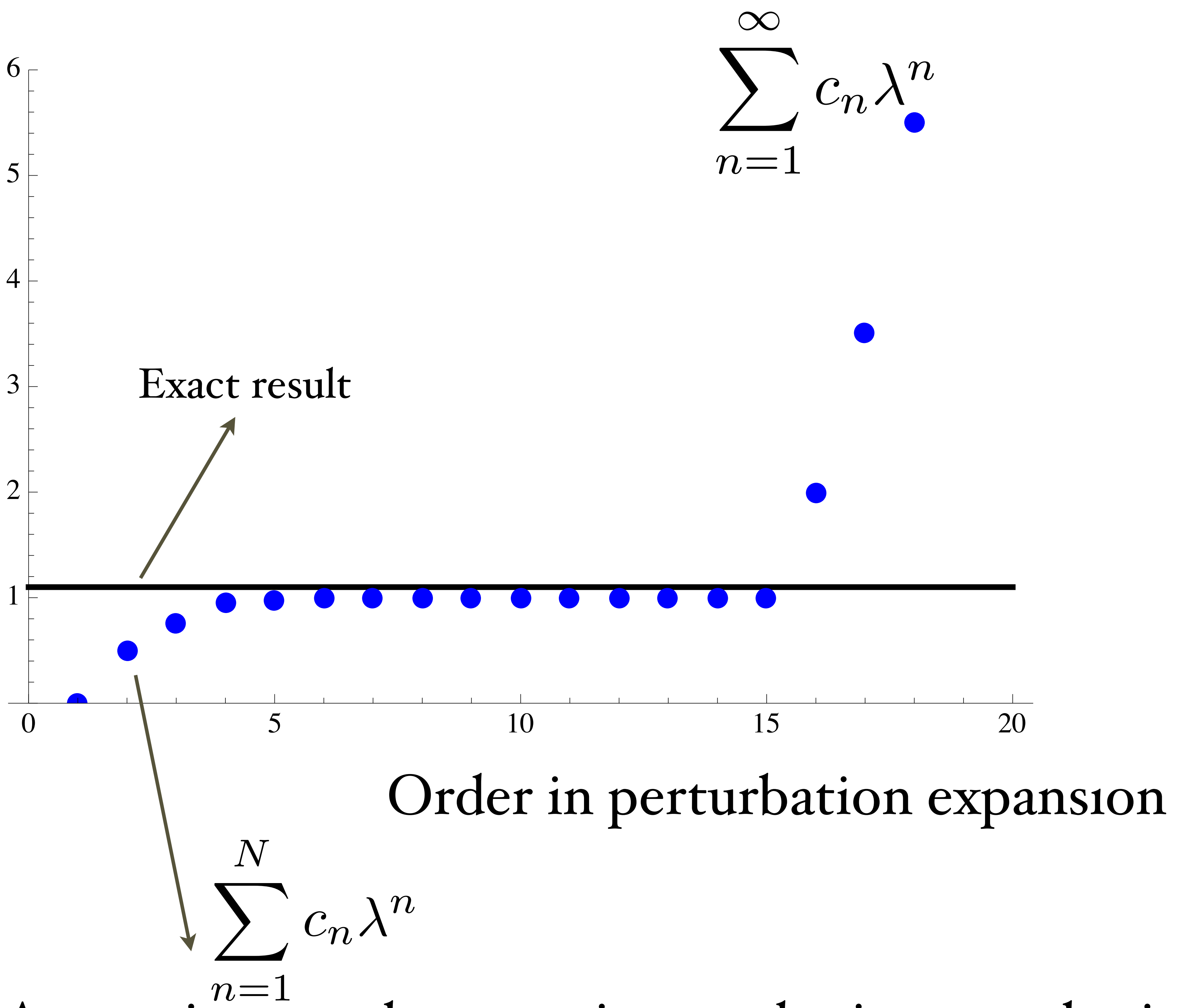
OUTLINE

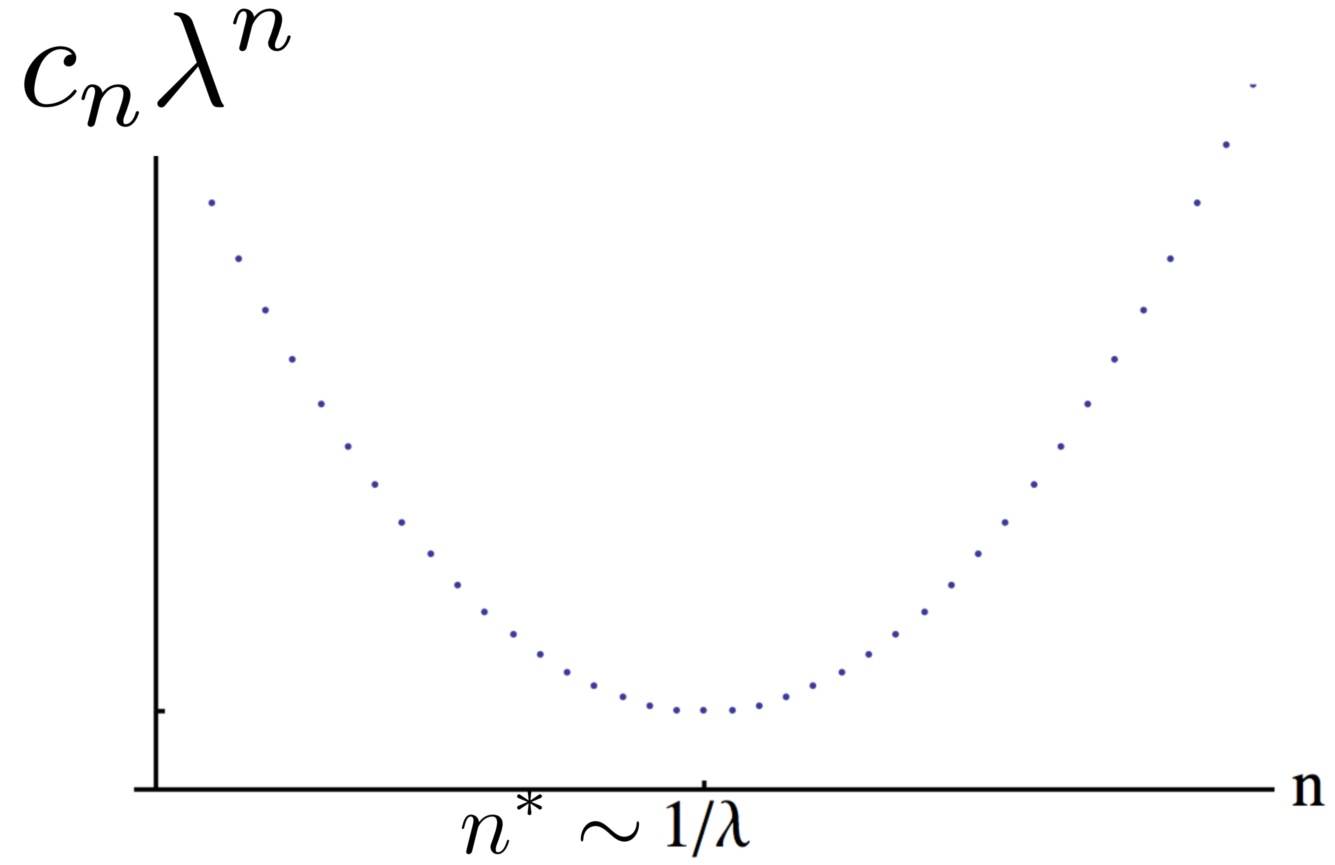
- General Structure of perturbation theory in simple terms
- Resurgence and Picard-Lefschetz theory in QM
- IR-Renormalon problem in QFT
- Adiabatic Continuity idea
- Reliable Semi-classics and fractional instantons on $R_3 \times S_1$
- TQFT coupling and a proposal for solution of IR-renormalon problem at strong coupling on large T_4

Universal behavior of perturbation theory

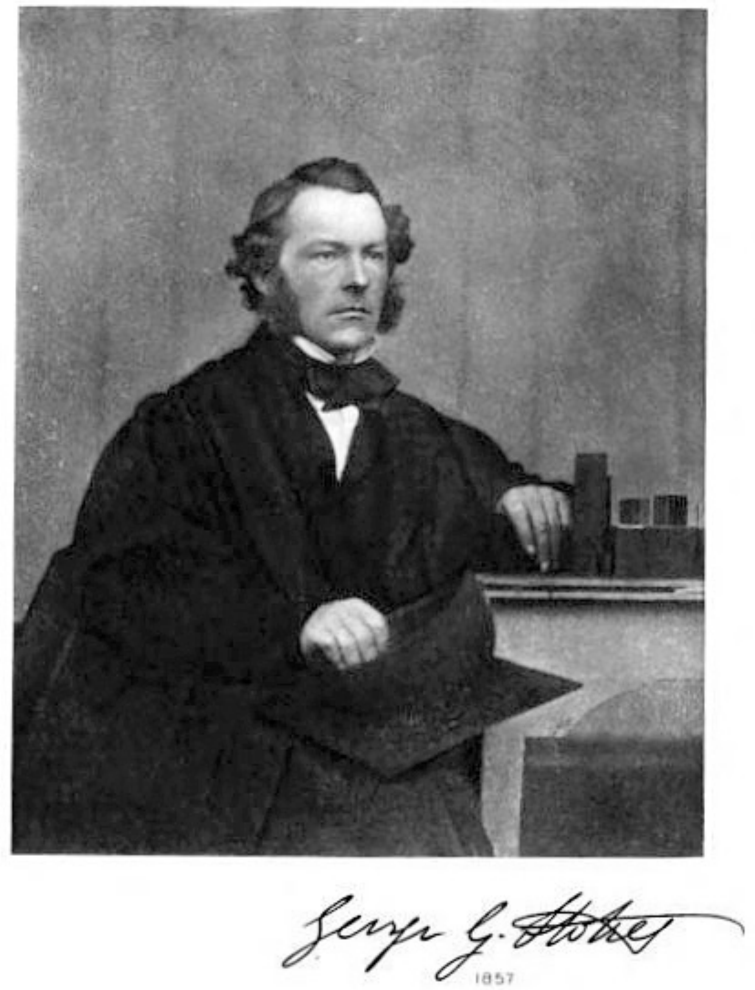


Approximate value at a given order in perturbative expansion

Traditional view on asymptotic series



Stokes (~1850s) brilliant realization:
There is an optimal order at which the error is minimized!



Watch carefully. This is important and easy.

Error: The deficit between exact result and the absolute best perturbation theory can do.

$$\text{Error} \sim n^*!\lambda^{n^*} \qquad \text{use Stirling} - \text{approximation}$$

$$\sim \left(\frac{n^*}{e}\right)^{n^*} \lambda^{n^*} \qquad \text{use} \qquad n^* \sim 1/\lambda$$

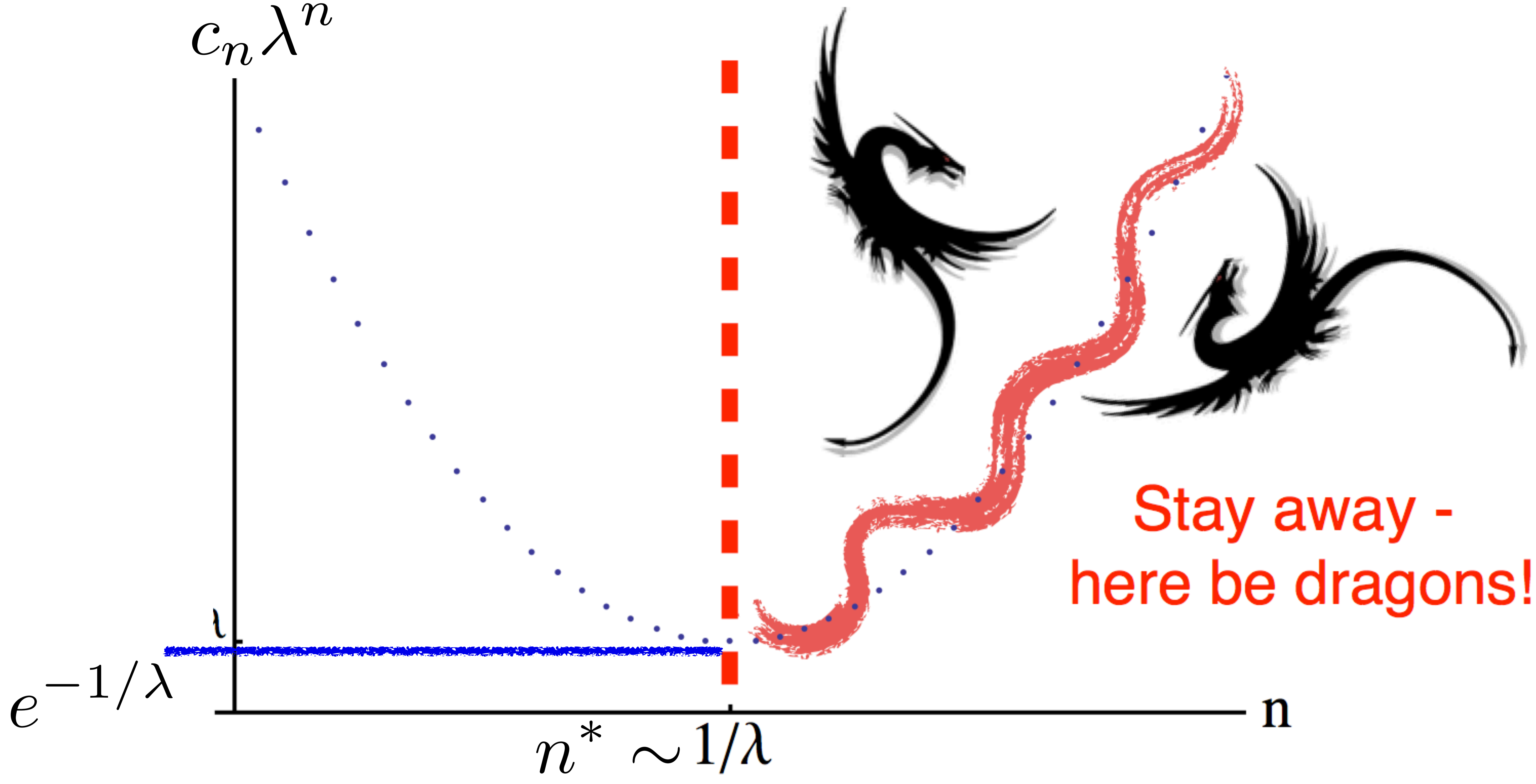
$$\sim e^{-1/\lambda}$$

Intrinsic (irremovable) error in perturbation theory is **non-perturbative!**

Exp[-1/λ] has essential singularity at zero, not describable in terms of pert. expansion.
If you try to do Taylor expansion, you obtain **0+0+0+0 ……** ad infinitum.

This is one reason why perturbative vs. non-perturbative phenomena in books are in different sections and not so much in relation to each other.

Traditional view on asymptotic series

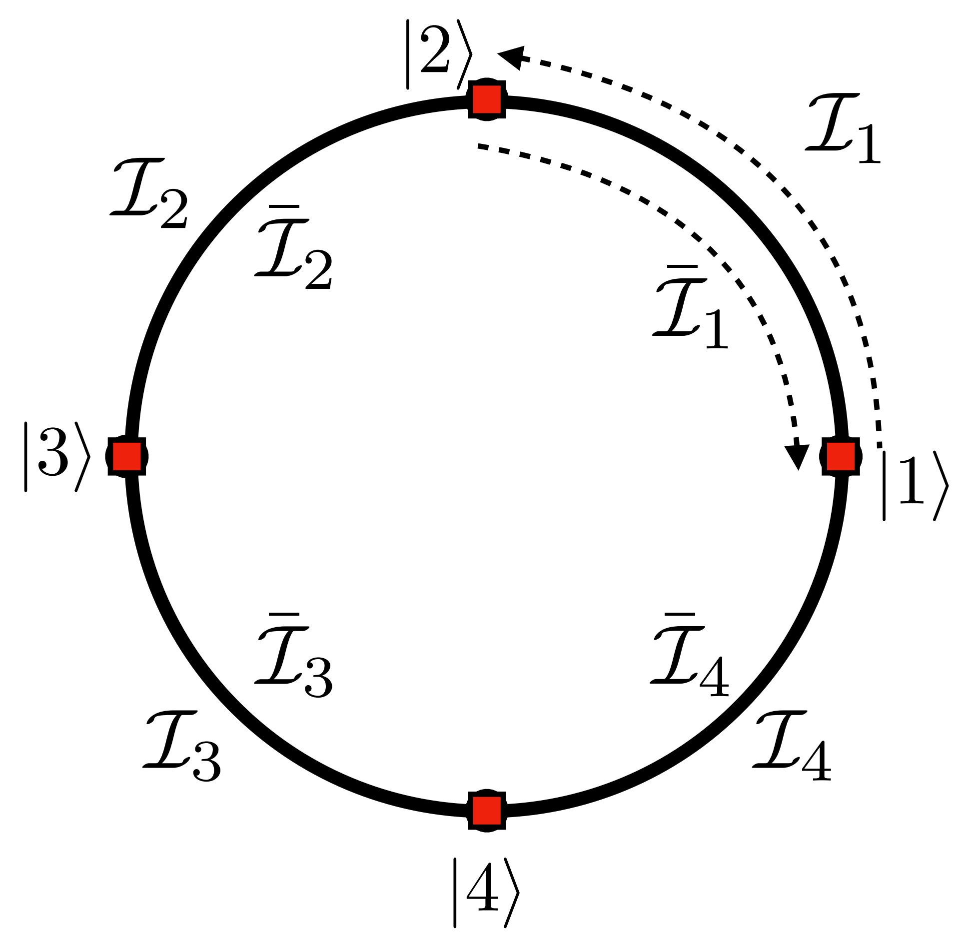


~1850, Stokes observed something even deeper. There is another saddle in the problem which contributes exactly as $\exp[-1/\lambda]$!

This is actually interesting. The intrinsic vagueness of perturbation theory is related to the existence of other saddles in the problem and its non-perturbative contribution!

I will describe this in an easy class of examples in QM.

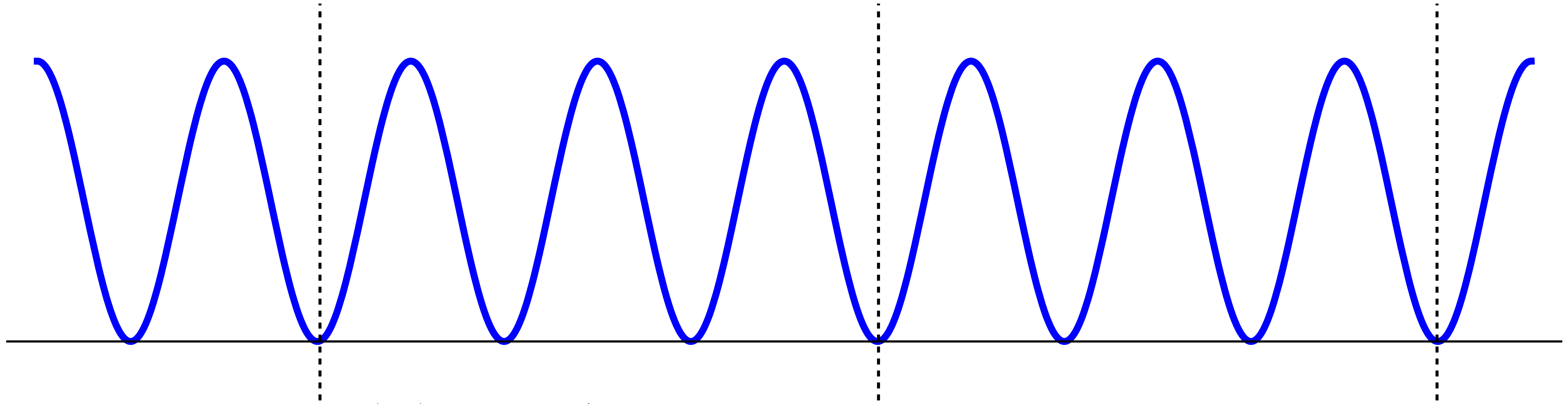
Example: QM of particle on a circle with periodic potential



$$V(q) = -\cos(Nq), \qquad q \sim q + 2\pi$$

$$\mathbb{Z}_N : q \mapsto q + \frac{2\pi}{N}$$

theta term : $\quad \mathrm{i}\frac{\theta}{2\pi}\int \mathrm{d}q$



$q(\tau) \in \mathbb{R}/2\pi N\mathbb{Z}, \;\; N = 4$

Making sense of semi-classical expansion in QM

$$f(\lambda\hbar) \sim \sum_{k=0}^{\infty} c_{(0,k)}\,(\lambda\hbar)^k + \sum_{n=1}^{\infty} (\lambda\hbar)^{-\beta_n}\, e^{-n\,A/(\lambda\hbar)} \sum_{k=0}^{\infty} c_{(n,k)}\,(\lambda\hbar)^k$$

pert. th. n-instanton factor pert. th. around n-instanton

All series appearing above are asymptotic, i.e., divergent as $c_{(0,k)} \sim k!$. The combined object is called trans-series.

Borel resummation idea: If $P(\lambda) \equiv P(g^2) = \sum_{q=0}^{\infty} a_q g^{2q}$ has convergent Borel transform

$$BP(t) := \sum_{q=0}^{\infty} \frac{a_q}{q!} t^q$$

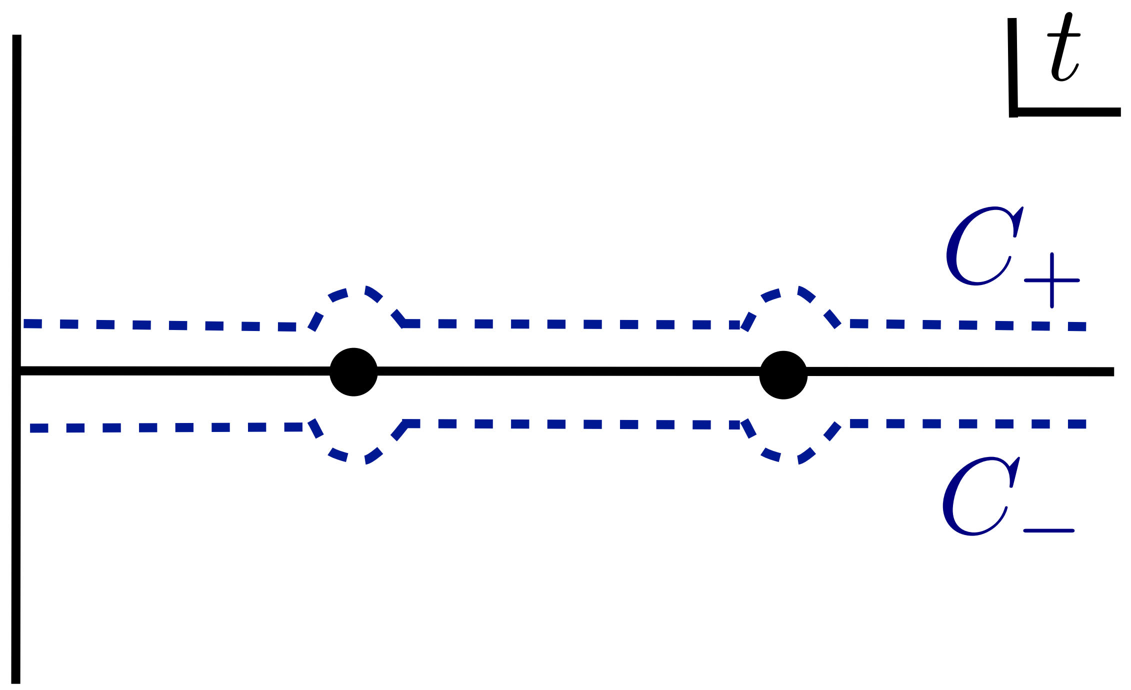
in neighborhood of $t = 0$, then

$$\mathbb{B}(g^2) = \frac{1}{g^2} \int_0^{\infty} BP(t) e^{-t/g^2} dt\ .$$

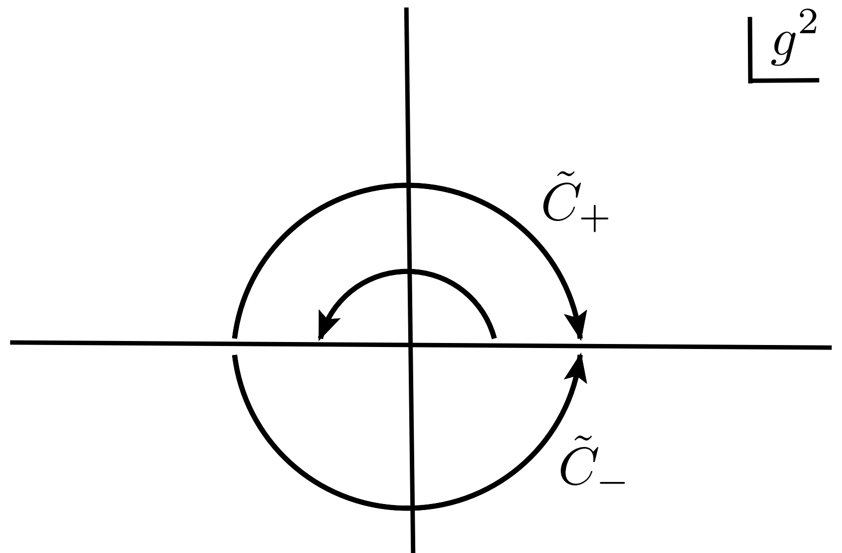
formally gives back $P(g^2)$, but is ambiguous if $BP(t)$ has singularities at $t \in \mathbb{R}^+$:

Perturbation theory: Borel plane, lateral Borel sums, ambiguity

Directional (sectorial) Borel sum. $\mathcal{S}_\theta P(g^2) \equiv \mathbb{B}_\theta(g^2) = \frac{1}{g^2}\int_0^{\infty\cdot e^{i\theta}} BP(t)\, e^{-t/g^2}\, dt$



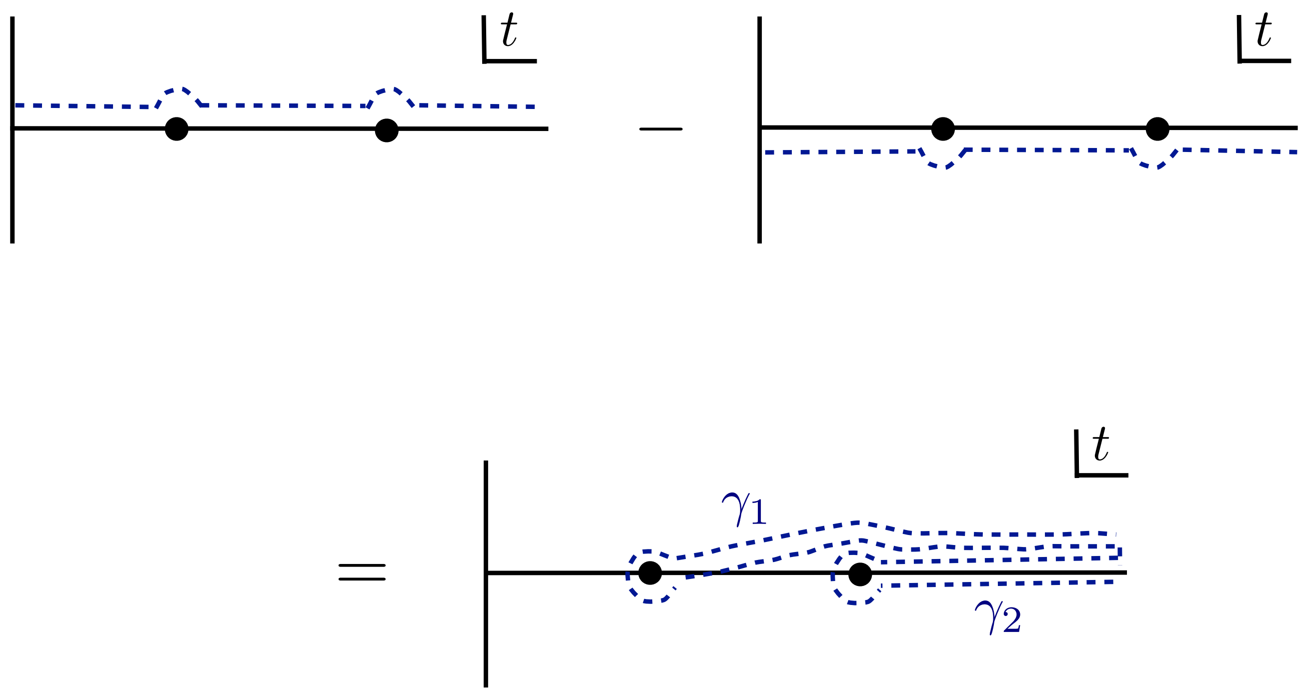
left and right Borel sums



Same structure by analytic continuation

$$\mathbb{B}_{0\pm}(|g^2|) = \mathrm{Re}\,\mathbb{B}_0(|g^2|) \pm i\,\mathrm{Im}\,\mathbb{B}_0(|g^2|), \qquad \mathrm{Im}\,\mathbb{B}_0(|g^2|) \sim e^{-2S_I} \sim e^{-2A/g^2}$$

The *non-equality* of the left and right Borel sum means the series is *non-Borel summable or ambiguous*. The ambiguity has the same form of a 2-instanton factor (not 1). The measure of ambiguity (Stokes automorphism/jump in g-space interpretation):



$$\mathcal{S}_{\theta+} = \mathcal{S}_{\theta-} \circ \underline{\mathfrak{S}}_\theta \equiv \mathcal{S}_{\theta-} \circ (\mathbf{1} - \mathrm{Disc}_{\theta-}),$$

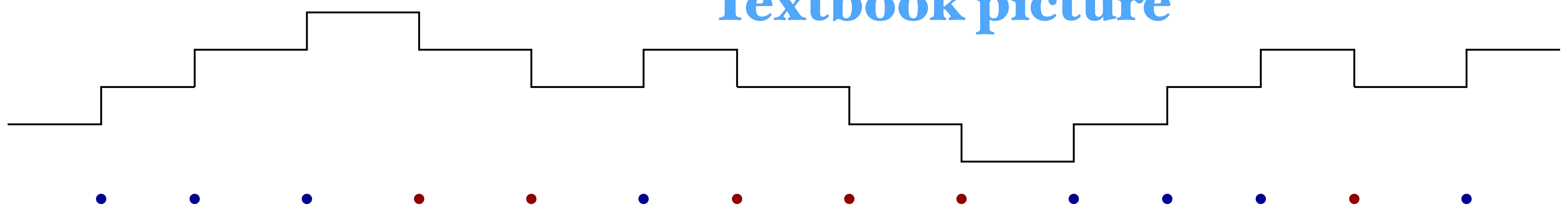
$$\mathrm{Disc}_{\theta-}\,\mathbb{B} \sim e^{-t_1/g^2} + e^{-t_2/g^2} + \dots \qquad t_i \in e^{i\theta}\mathbb{R}^+$$

Resurgence: Ecalle, 80s

Instantons and Bogomolny--Zinn-Justin (BZJ) prescription

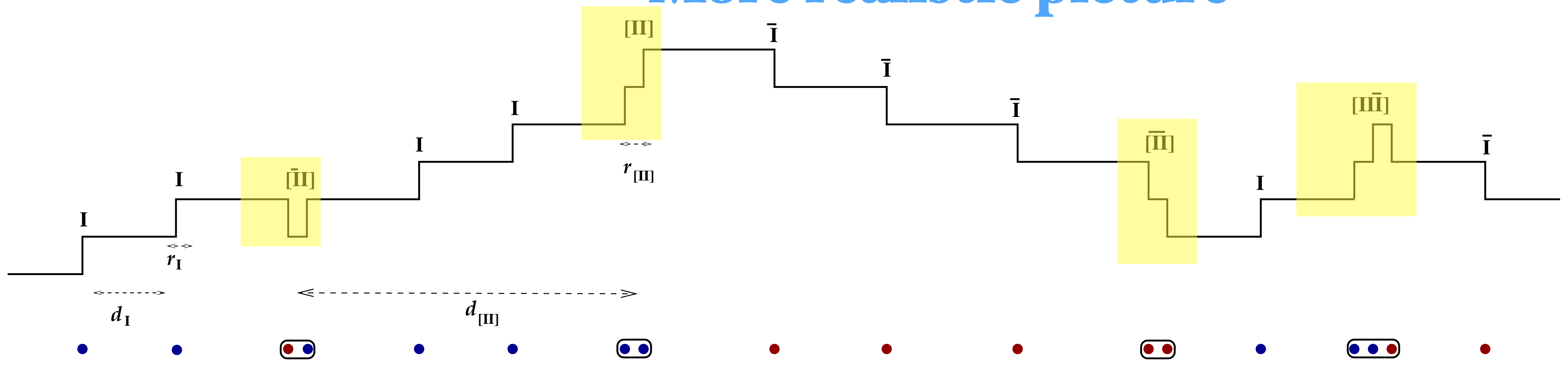
BZJ, QM (80s): for double well potential,
Here, I show it for a periodic potential.
Dilute instanton, molecular instanton gas.

Textbook picture



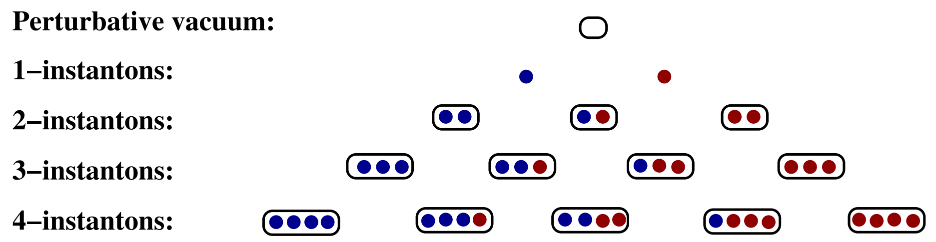
a) Dilute gas of 1–instantons

More realistic picture



b) Dilute gas of 1–instantons, 2–instantons, and other molecular–events

$$
\begin{array}{ccccccc}
r_{\rm I} & \ll & r_{[\rm II]} \sim \ell_{\rm qzm} & \ll & d_{\rm I} & \ll & d_{[\rm II]}, \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
L & \ll & L\,\log\left(\frac{1}{g^2}\right) & \ll & L\,e^{S_0} & \ll & L\,e^{2S_0}.
\end{array}
$$



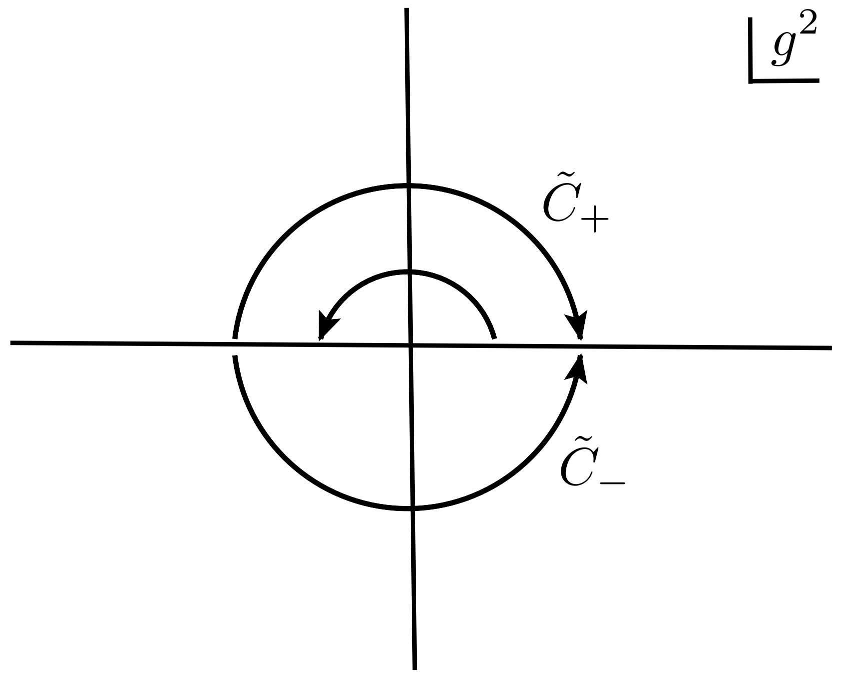
c) Representatives of n–instanton events, sketched according to the resurgence triangle.

How to make sense out of correlated events?

$[\mathcal{I}\mathcal{I}]$ Evaluate quasi-zero mode integral.Easy.

$[\mathcal{I}\bar{\mathcal{I}}]$ Naive calculation meaningless* at $g^2 > 0$.
The quasi-zero mode integral is dominated at small-separations where a molecular instanton is meaningless. BZJ: Continue to $g^2 < 0$, evaluate there, and continue back to $g^2 > 0$: two fold-ambiguous!

$$[\mathcal{I}\bar{\mathcal{I}}]_{\theta=0^\pm} = \mathrm{Re}\,[\mathcal{I}\bar{\mathcal{I}}] + i\,\mathrm{Im}\,[\mathcal{I}\bar{\mathcal{I}}]_{\theta=0^\pm}$$



*: Retrospectively, it better be so, because we are on a Stokes line.

Instanton interactions

Since instanton equations and Euclidean eq of motion are non-linear, two instanton configurations is not a solution at finite separation.

$$x_{\mathcal{I}\mathcal{I}}(\tau) = x_{\mathcal{I}}(\tau - \tau_1) + x_{\mathcal{I}}(\tau - \tau_2),$$
$$x_{\mathcal{I}\bar{\mathcal{I}}}(\tau) = x_{\mathcal{I}}(\tau - \tau_1) - x_{\mathcal{I}}(\tau - \tau_2),$$

$$S_{\mathcal{I}\mathcal{I}}(\tau_{12}) = 2S_I + \frac{A}{g} e^{-\tau_{12}}, \qquad \text{repulsive},$$
$$S_{\mathcal{I}\bar{\mathcal{I}}}(\tau_{12}) = 2S_I - \frac{A}{g} e^{-\tau_{12}}, \qquad \text{attractive}$$

Attractive/repulsive are just words, inheritance from old literature. Caused too much confusion in past. This formula just means that these combos are not exact solution for finite separations. That is all. Tau direction is called quasi-moduli space.

Cluster expansion

Assume S^1_β with 1-minimum on it, $N = 1$

In the $\beta \to \infty$ limit, we can write Z as

$$Z = e^{-\beta E_0 P_0(g)} \left(1 + \frac{\xi}{1!} \int d\tau_1 + \frac{\xi^2}{2!} \int d\tau_1 d\tau_2 \; e^{-V_{12}} + \frac{\xi^3}{3!} \int d\tau_1 d\tau_2 d\tau_3 \; e^{-V_{123}} + \ldots \right) .$$

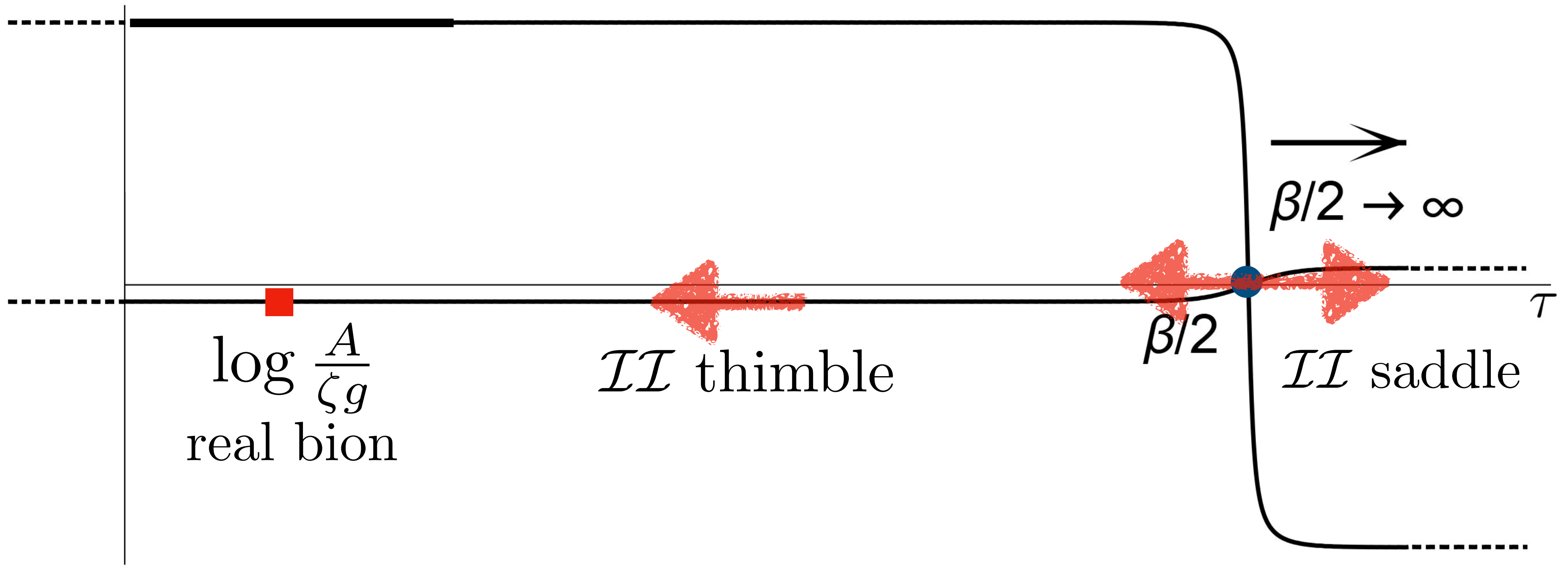
whee $\xi \sim e^{-S_I}$ is the instanton amplitude.

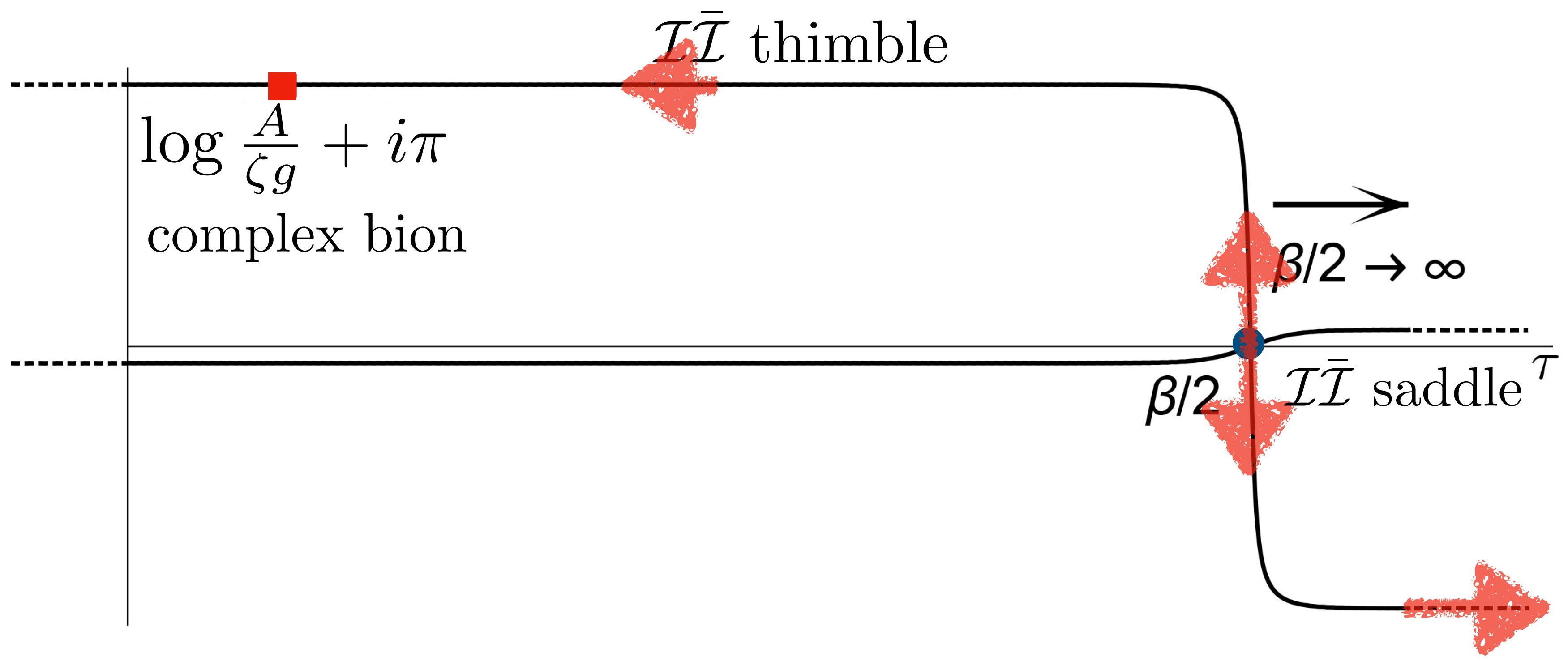
$$\begin{aligned} Z_{\text{dilute}} &= e^{-\beta\left(E_0 P_0(g) - [\mathcal{I}] - [\bar{\mathcal{I}}] - [\mathcal{I}^2] - [\bar{\mathcal{I}}^2] - [\mathcal{I}\bar{\mathcal{I}}]_\pm - [\bar{\mathcal{I}}\mathcal{I}]_\pm - [\mathcal{I}^3] - [\mathcal{I}^2\bar{\mathcal{I}}] \ldots\right)} \\ &= e^{-\beta E_0 P_0(g)} \left(\sum_{n_{\mathcal{I}}=0}^{\infty} \frac{\beta^{n_{\mathcal{I}}} [\mathcal{I}]^{n_{\mathcal{I}}}}{n_{\mathcal{I}}!} \right) \left(\sum_{n_{\bar{\mathcal{I}}}=0}^{\infty} \frac{\beta^{n_{\bar{\mathcal{I}}}} [\mathcal{I}]^{n_{\bar{\mathcal{I}}}}}{n_{\bar{\mathcal{I}}}!} \right) \left(\sum_{n_{\mathcal{I}\bar{\mathcal{I}}}=0}^{\infty} \frac{\beta^{n_{\mathcal{I}\bar{\mathcal{I}}}} [\mathcal{I}\bar{\mathcal{I}}]_\pm^{n_{\mathcal{I}\bar{\mathcal{I}}}}}{n_{\mathcal{I}\bar{\mathcal{I}}}!} \right) \ldots \end{aligned}$$

Steepest descent cycle for I-I and I-Ibar quasi-zero mode directions

Compactify $\mathbb{R} \to S^1_\beta$ in order to study $Z(\beta) = \mathrm{Tr}\,[e^{-\beta H}]$.
The interaction between two events is modified in a fairly obvious way into:

$$S(\tau) = \pm \frac{A}{g}\left(e^{-\tau} + e^{-(\beta-\tau)}\right)$$





Steepest descent cycle of $I\bar{I}$ critical point at infinity lives in the complex domain.

Borel-Ecalle summability in QM

$$[\mathcal{I}\bar{\mathcal{I}}]_{\pm} = \left(\mp i\pi - \gamma - \log\left(\frac{A}{g}\right) + \dots\right)[\mathcal{I}][\bar{\mathcal{I}}]$$

$$[\mathcal{I}\bar{\mathcal{I}}]_{\pm} \sim \left(\mp i\pi - \gamma - \log\left(\frac{A}{g}\right) + \dots\right) e^{-(2S_I)/g}\left(1 - \frac{5}{2}\cdot g - \frac{13}{8}\cdot g^2 \dots\right)$$

$$a_n(N=0) \sim -\frac{1}{\pi}\frac{n!}{(2S_I)^{n+1}}\left(1 - \frac{5}{2}\cdot\frac{(2S_I)^1}{n} - \frac{13}{8}\cdot\frac{(2S_I)^2}{n(n-1)} + \dots\right)$$

$$\mathrm{Im}\,\mathbb{B}_{0,\theta=0^{\pm}} + \mathrm{Im}\,[\mathcal{I}\bar{\mathcal{I}}]_{\theta=0^{\pm}} = 0\,, \qquad \text{up to } O(e^{-4S_I})$$

The leading terms (structures) obtained in Bogomolny and Zinn-Justin early 80s, but not sufficiently appreciated. The interesting thing is, B-ZJ was not an unknown work. The problem was that their methods in the derivation did not convince people. I was personally fascinated by what they did, and was convinced that their main claim was correct.

The overall structure was obtained in 2014, in Gerald Dunne and MU.

Borel-Ecalle summability

- periodic potential: $V(x) = \frac{1}{g^2}\sin^2(g\,x)$
- vacuum saddle point

$$c_n \sim n!\left(1 - \frac{5}{2}\cdot\frac{1}{n} - \frac{13}{8}\cdot\frac{1}{n(n-1)} - \dots\right)$$

- instanton/anti-instanton saddle point:

$$\mathrm{Im}\,E \sim \pi\, e^{-2\frac{1}{2g^2}}\left(1 - \frac{5}{2}\cdot g^2 - \frac{13}{8}\cdot g^4 - \dots\right)$$

- double-well potential: $V(x) = x^2(1-gx)^2$
- vacuum saddle point

$$c_n \sim 3^n n!\left(1 - \frac{53}{6}\cdot\frac{1}{3}\cdot\frac{1}{n} - \frac{1277}{72}\cdot\frac{1}{3^2}\cdot\frac{1}{n(n-1)} - \dots\right)$$

- instanton/anti-instanton saddle point:

$$\mathrm{Im}\,E \sim \pi\, e^{-2\frac{1}{6g^2}}\left(1 - \frac{53}{6}\cdot g^2 - \frac{1277}{72}\cdot g^4 - \dots\right)$$

resurgence: fluctuations about the instanton/anti-instanton saddle are determined by those about the vacuum saddle. (Generalization of Berry-Howls to infinite dimensional integrals)

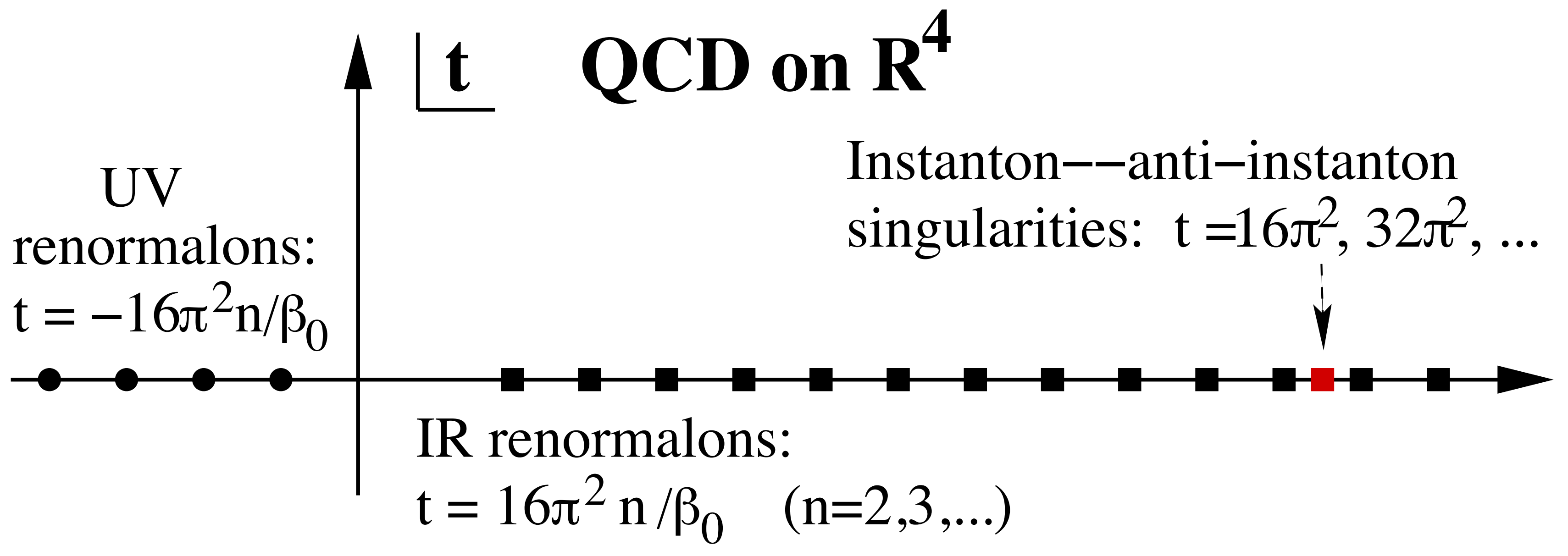
Can this work in QFT? QCD on R4 or CP(N-1) on R2?

't Hooft(79) :***No,*** *on R4,* Argyres, MÜ: ***Yes,*** *on R3 x S1,*
F. David(84), Beneke(93) : ***No,*** *on R2.* Dunne, MÜ: ***Yes,*** *on R1 x S1*

Why doesn't it work?

Instanton-anti-instanton contribution, calculated in some way, gives an ±i exp[-2S_I].
Lipatov(77): Borel-transform BP(t) has singularities at t_n= 2n g^2 S_I. (Modulo the standard IR problems with 2d instantons, also see Bogomolny-Fateyev(77)).

BUT, BP(t) has other (more important) singularities closer to the origin of the Borel-plane. (not due to factorial growth of number of diagrams!)



't Hooft called these IR-renormalon singularituies with the hope/expectation that they would be associated with a saddle point like instantons

.

No such configuration is known!!
A real problem in QFT, means pert. theory, as is, ill-defined. How to cure starting from micro-dynamics?

Standard lore in Yang-Mills

$$t^{\mathbb{R}^4}_{[I\bar{I}]} = 2S_I g^2 = 16\pi^2$$ Instanton (I-Ibar) singularity

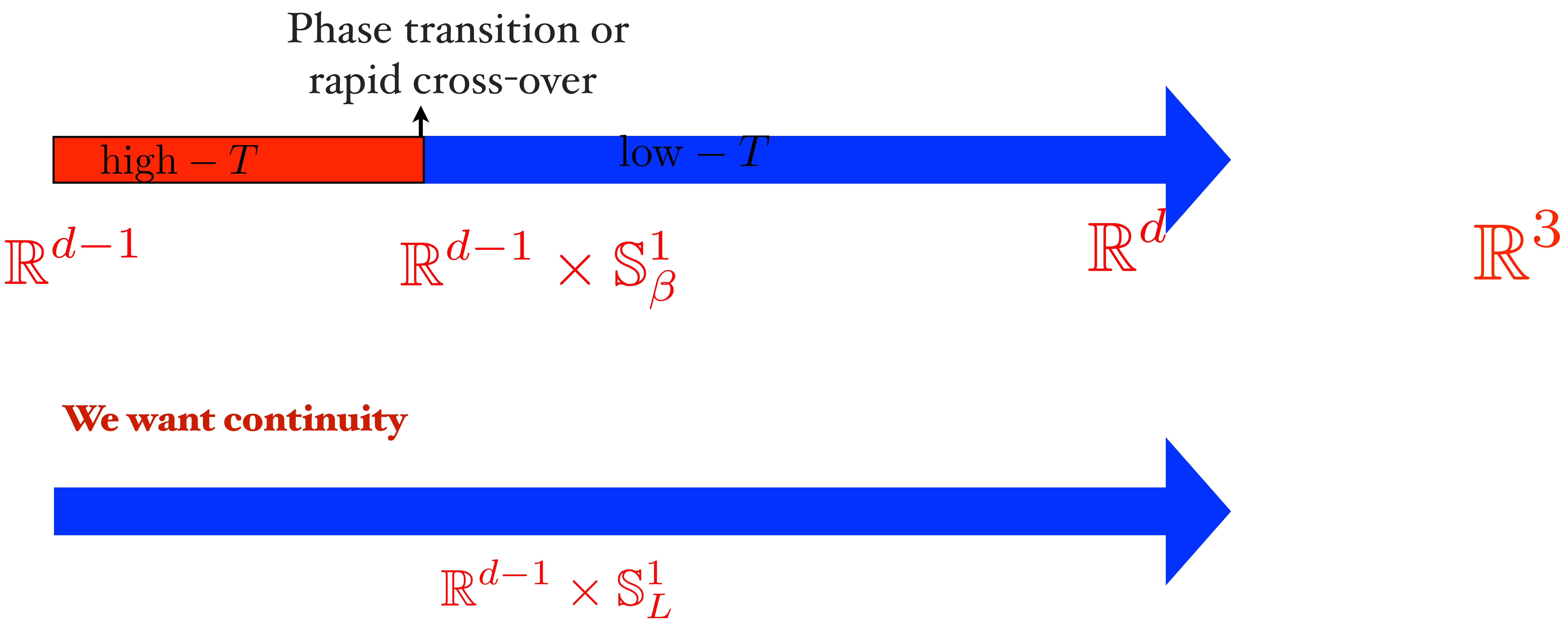
$$t^{\mathbb{R}^4}_{\text{ren.}} = \frac{4}{\beta_0} S_I g^2 = \frac{2}{\beta_0} t^{\mathbb{R}^4}_{[I\bar{I}]} = \frac{6}{11N} t_{[I\bar{I}]},$$ Leading renormalon singularity in YM

'tHooft called these IR-renormalon singularity with the hope/expectation that they would be associated with a saddle point like instantons!

Change the Question: What happens if we can make the most interesting QFTs semi-classically calculable? Yang-Mills, CP(N-1), Supersymmetric Yang-Mills,..

Is this even possible?

Adiabatic continuity idea



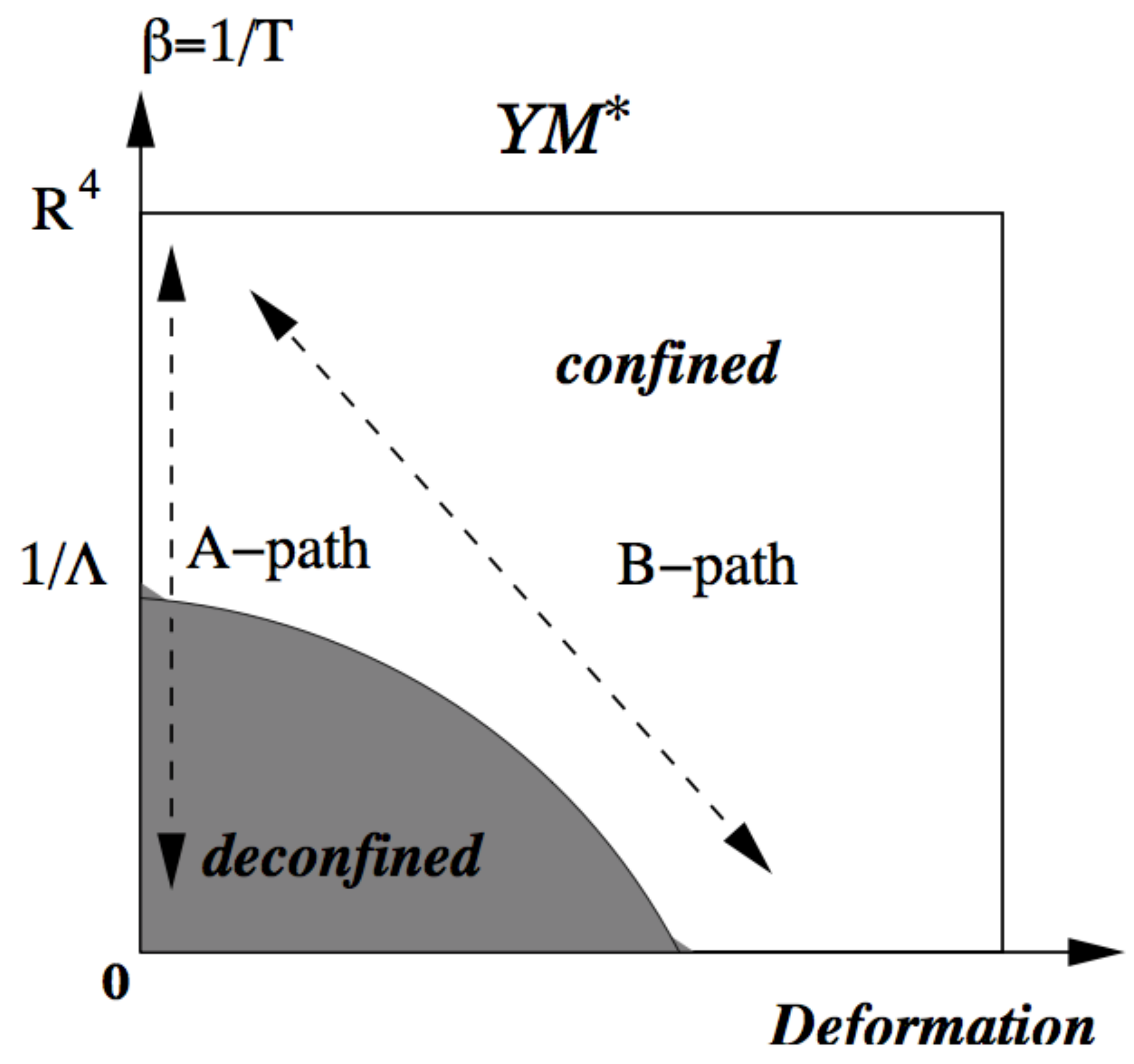
Thermal finite-N: Rapid crossover at strong scale
Thermal large-N: Sharp phase transition at strong scale

Prevent both by using circle compactification or double-trace deformation.

Semi-classical Calculablity In Yang-Mills on $\mathbb{R}^3 \times S^1$

$$S^{\mathrm{YM}^*} = S^{\mathrm{YM}} + \int_{R^3 \times S^1} P[\Omega(\mathbf{x})] \qquad P[\Omega] = A\frac{2}{\pi^2 L^4} \sum_{n=1}^{\lfloor N/2 \rfloor} \frac{1}{n^4} \left|\mathrm{tr}\,(\Omega^n)\right|^2$$

- Double-trace deformation that prevents center-breaking and admits a semi-classically calculable regime connected to R4 limit (Yaffe, MU, 2008).

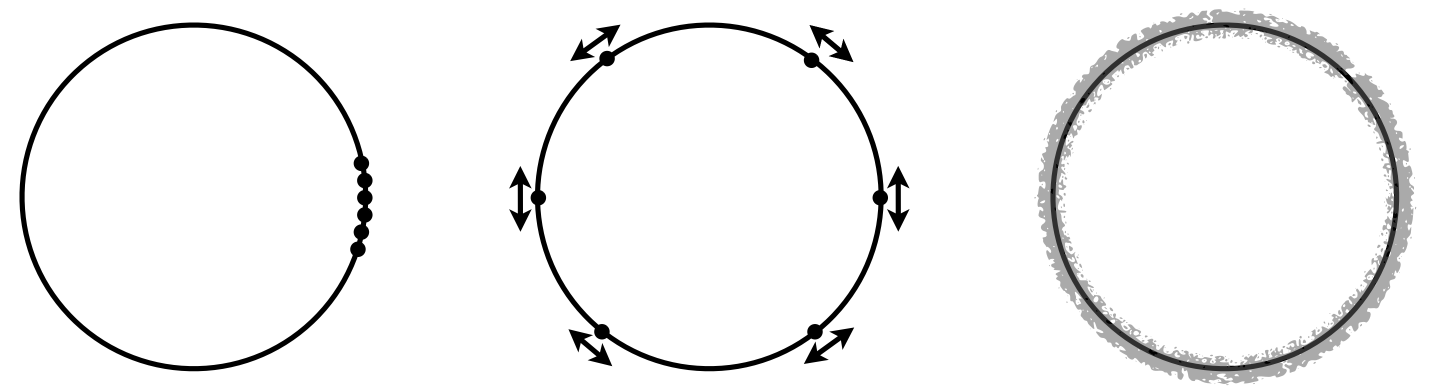


Abelianization and abelian duality

$$SU(N) \to U(1)^{N-1}$$

Similar to Polyakov model in 3d (1974) and Seiberg-Witten in 4d (1994), dynamics abelianize, but via a compact group valued field

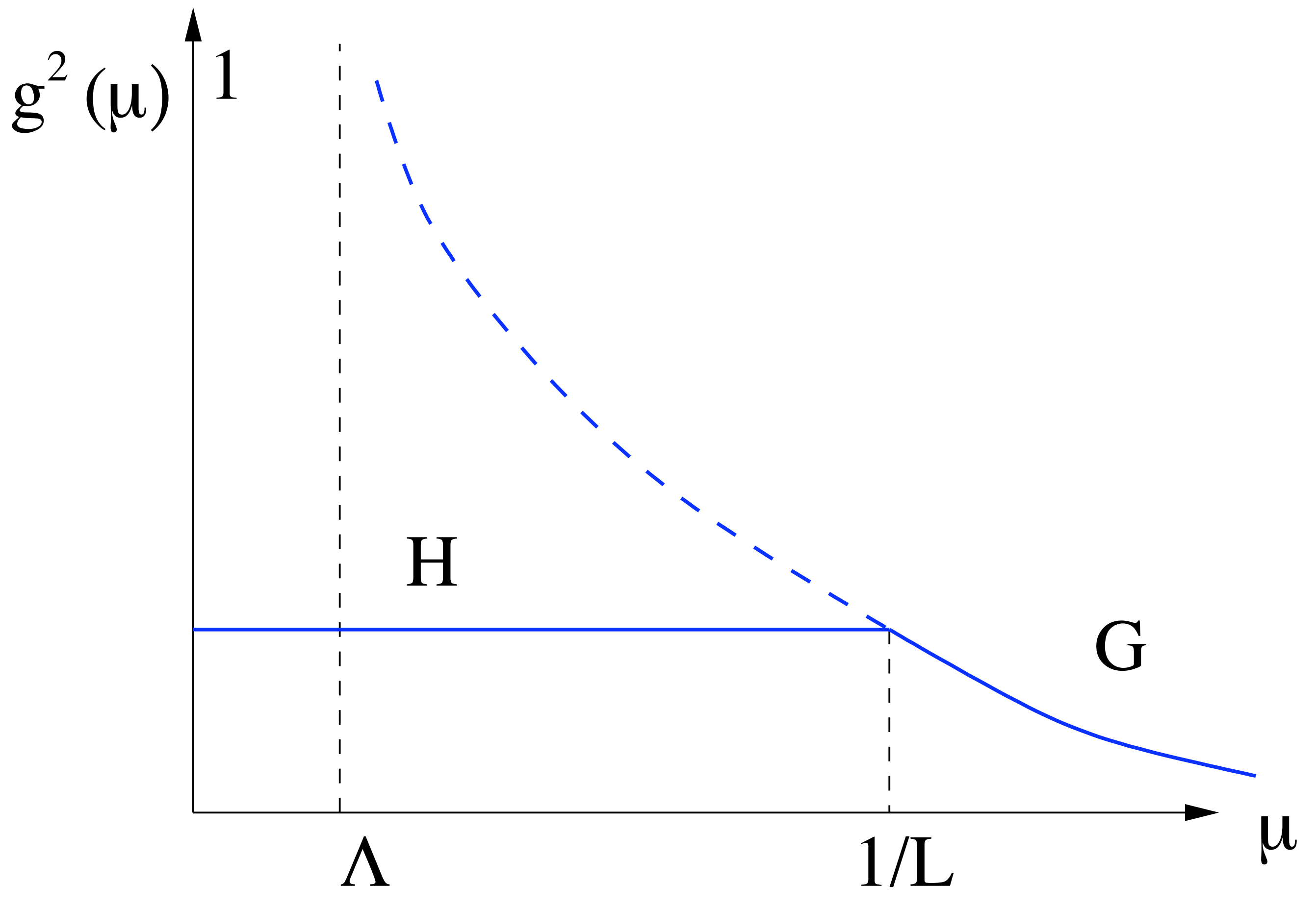
Three types of holonomy



(a) center broken

(b) center-stable weak coupling

(c) center-stable strong coupling



$$L = \frac{1}{4} F_{\mu\nu}^2 \longleftrightarrow \frac{1}{2} (\partial_\mu \sigma)^2$$

Gapless to all orders in perturbation theory.
How about NP-effects?

Topological configurations: Monopole-instantons

1-defects, Monopole-instantons: Associated with the N-nodes of the affine Dynkin diagram of SU(N) algebra.
The Nth type corresponds to the affine root and is present only because the theory is *locally 4d*!

van Baal, Kraan, (97/98),
Lee-Yi, Lee-Lu (97)

$$\mathcal{M}_k \sim e^{-S_k} e^{-\alpha_k \cdot b + i\alpha_k \sigma + i\theta/N}, \qquad k = 1, \ldots, N$$

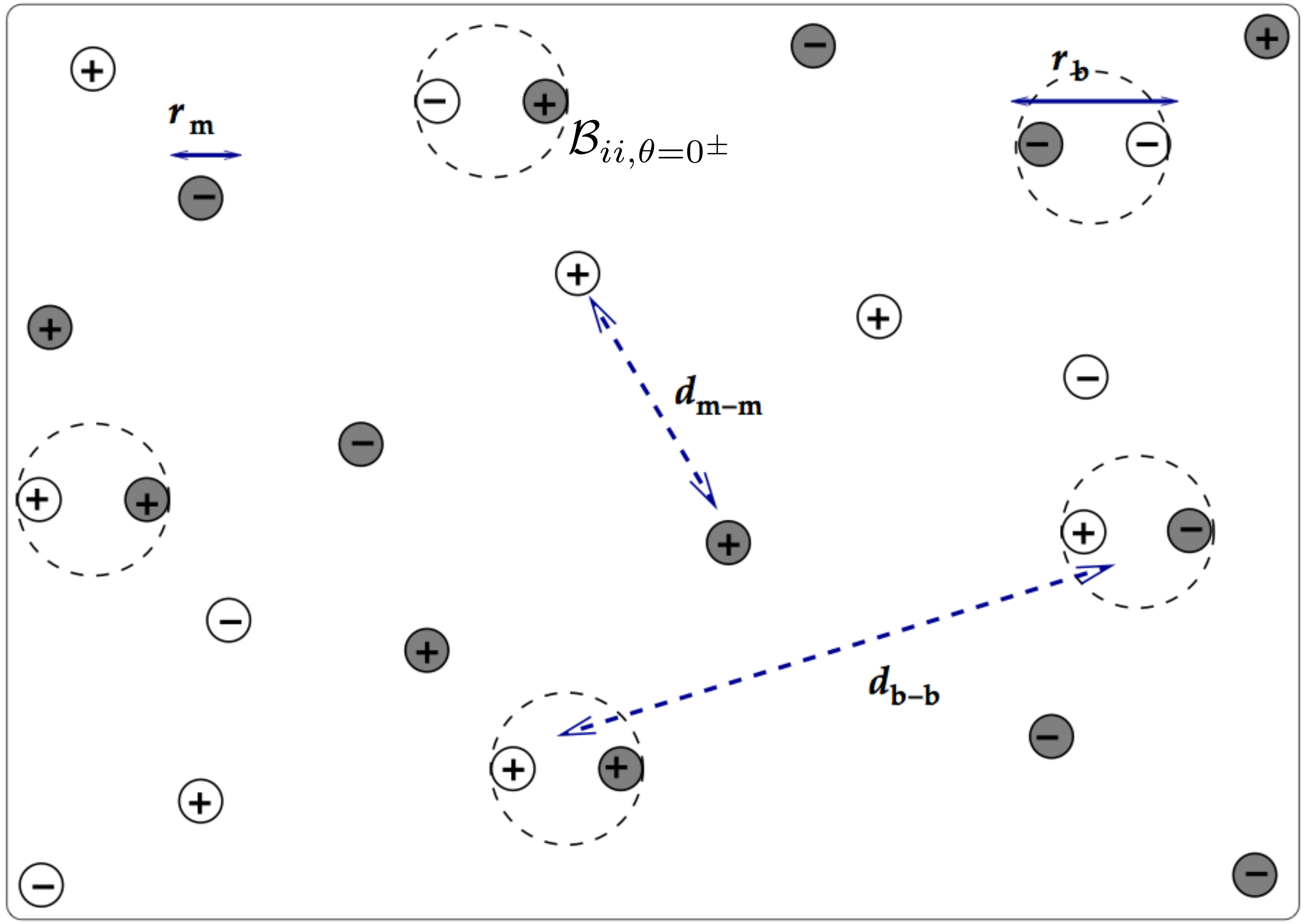
$$S_k = \frac{8\pi^2}{g^2 N} = \frac{S_I}{N}$$

Action 1/N of the 4d instanton, keep this in mind!

Proliferation of monopole-instantons generates a non-perturbative mass gap for gauge fluctuations, similar to 3d Polyakov model (Polyakov, 74). It is first generalization thereof to local 4d theory!

Deformed YM, Euclidean vacuum

But interestingly, correlated monopole-anti-monopole event is 2-fold ambiguous as in QM.



Relation to R4? Will comment on this later...

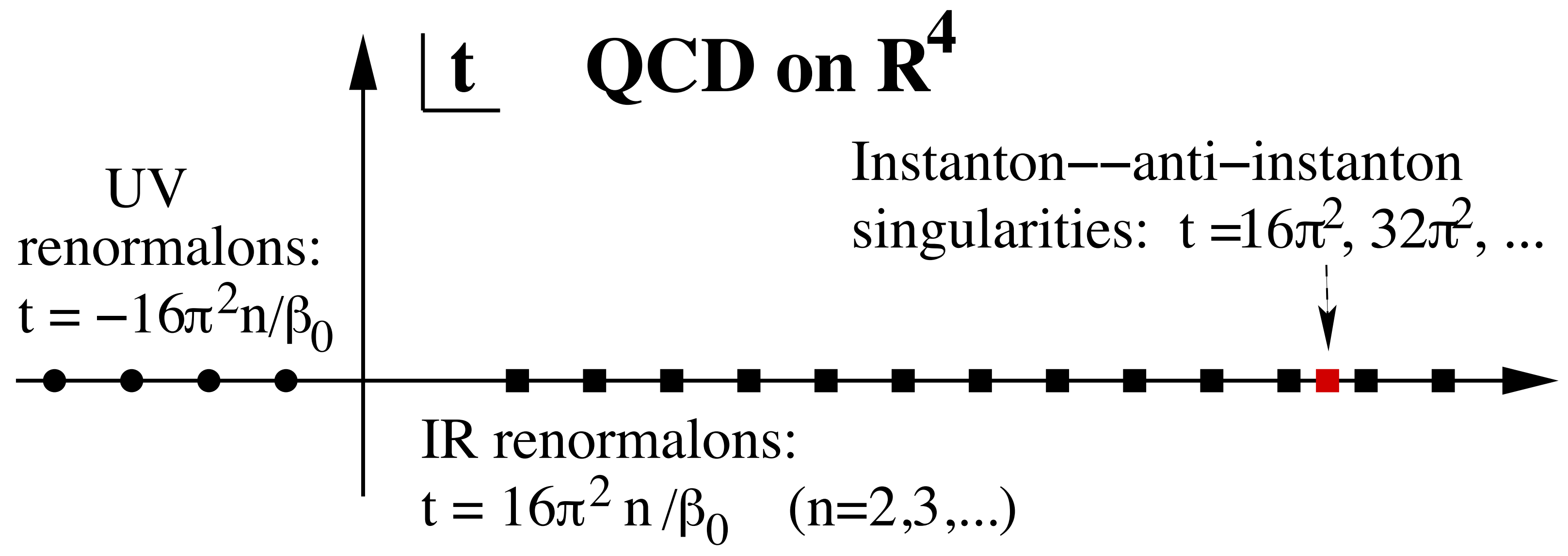
$$\langle F^2 \rangle_{0^\pm} \propto \mathcal{M}_i + [\mathcal{M}_i \bar{\mathcal{M}}_j] + [\mathcal{M}_i \bar{\mathcal{M}}_i]_{0^\pm} + \dots$$

$$t^* = \frac{1}{N} 2S_I = \frac{1}{N} t_{[I\bar{I}]}.$$

Ambiguity in condensate sourced by neutral bion.

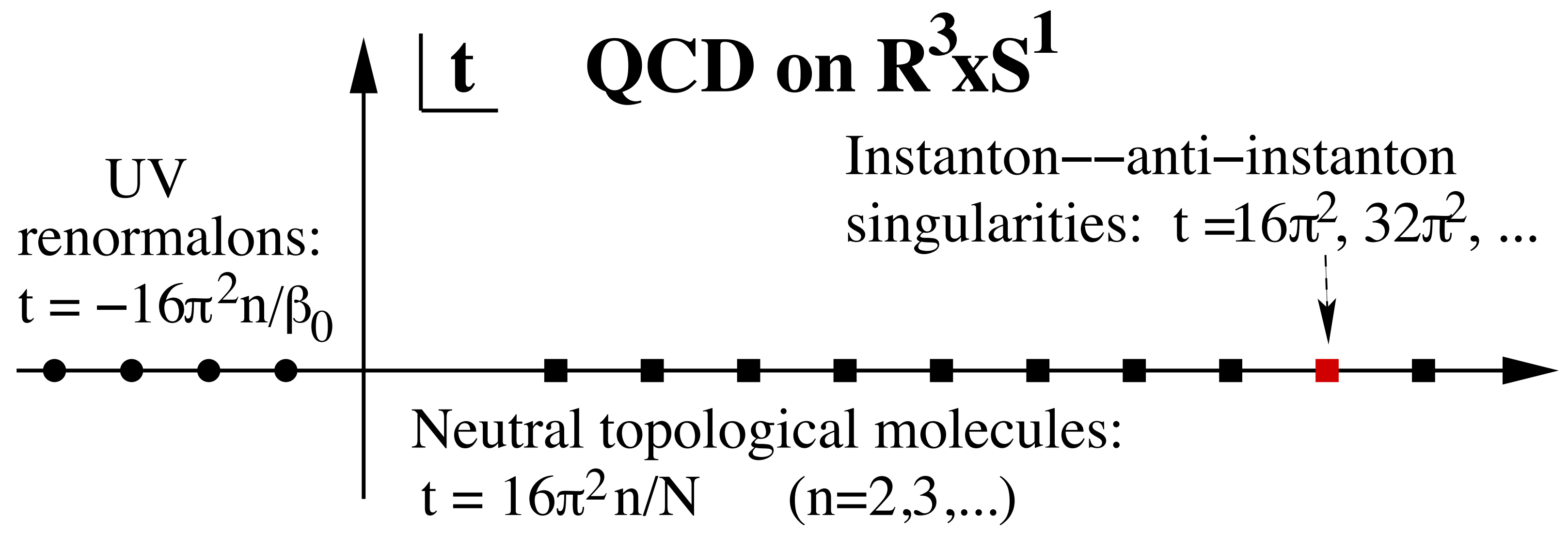
Semi-classical renormalons as neutral bions

Claim (w/ Argyres in 4d) and (w/ Dunne in 2d): Neutral bions and neutral topological molecules are semi-classical avatars of 't Hooft's elusive renormalons.
We showed in 2d sigma models, but it is conjectural in 4d.



$$t^{\mathbb{R}^4}_{\rm ren.} = \frac{2}{\beta_0} t^{\mathbb{R}^4}_{[I\bar{I}]} = \frac{6}{11N} t_{[I\bar{I}]},$$

?



$$t^* = \frac{1}{N} 2S_I = \frac{1}{N} t_{[I\bar{I}]}.$$

We were content with the fact that we found singularities at order 1/N of the instanton singularity. It was good enough for us given the history of the subject.

But a number objections appeared recently since the two positions do not agree precisely...

Is the semi-classical result and the surmised result on R_4 in contradiction?

To answer this question, we have to be able to carry over some results that are reliable in semi-classical domain outside the region of validity of semi-classics.

Does not sound easy… What can we do?

Here comes the TQFT and 't Hooft flux backgrounds.

TQFT coupling in Yang-Mills

or can 't Hooft help 't Hooft?

or can 't Hooft who introduced discrete-flux backgrounds help 't Hooft who introduced IR renormalon problem?

Coupling Z_N TQFT to YM-formally

To turn on a classical background gauge field for the $\mathbb{Z}_N^{[1]}$ 1-form symmetry, introduce pair of $U(1)$ 2-form and 1-form gauge fields $(B^{(2)}, B^{(1)})$ satisfying

$$NB^{(2)} = \mathrm{d}B^{(1)}, \qquad N\int B^{(2)} = \int \mathrm{d}B^{(1)} = 2\pi\mathbb{Z}$$

Promote $SU(N)$ gauge field to a $U(N)$: $\widetilde{a} = a + \frac{1}{N}B^{(1)}$

Kapustin, Seiberg, 2014,
Komargodski et.al. 2017

$$S[B^{(2)}, B^{(1)}, \widetilde{a}] = \frac{1}{2g_{\mathrm{YM}}^2}\int \mathrm{tr}[(\widetilde{F} - B^{(2)}) \wedge \star(\widetilde{F} - B^{(2)})] + \frac{\mathrm{i}\,\theta_{\mathrm{YM}}}{8\pi^2}\int \mathrm{tr}[(\widetilde{F} - B^{(2)}) \wedge (\widetilde{F} - B^{(2)})]$$

Modified instanton equation: $(\widetilde{F} - B^{(2)}) = \mp \star (\widetilde{F} - B^{(2)})$

Action: $$S = \mp \frac{8\pi^2}{g^2} \frac{1}{8\pi^2} \int \mathrm{tr}[(\widetilde{F} - B^{(2)}) \wedge (\widetilde{F} - B^{(2)})] = \frac{S_I}{N}$$

because $$\frac{N}{8\pi^2} \int B^{(2)} \wedge B^{(2)} \in \frac{1}{N}\mathbb{Z}$$

In SU(N) theory coupled to Z_N background gauge field, the configurations which satisfy BPS bound have action S_I/N, just like our monopole-instantons on R_3 x S_1.

This is true both at weak and strong coupling!

SU(N) and PSU(N) theories are locally the same. But bundles are different.

$$Z_{SU(N)} = \sum_{W\in\mathbb{Z}} e^{\mathrm{i}\theta W} Z_W$$

$$Z_{SU(N)}(\ell, m) = \sum_{W\in\mathbb{Z}} e^{\mathrm{i}\theta\left(W+\frac{(\ell\cdot m)}{N}\right)} Z_W(\ell, m)$$

$$Z_{PSU(N)_p} = \sum_{\substack{W\in\mathbb{Z} \\ \ell, m\in(\mathbb{Z}_N)^3}} e^{\mathrm{i}\frac{2\pi}{N}p\,(\ell\cdot m)} e^{\mathrm{i}\theta\left(W+\frac{(\ell\cdot m)}{N}\right)} Z_W(\ell, m)$$

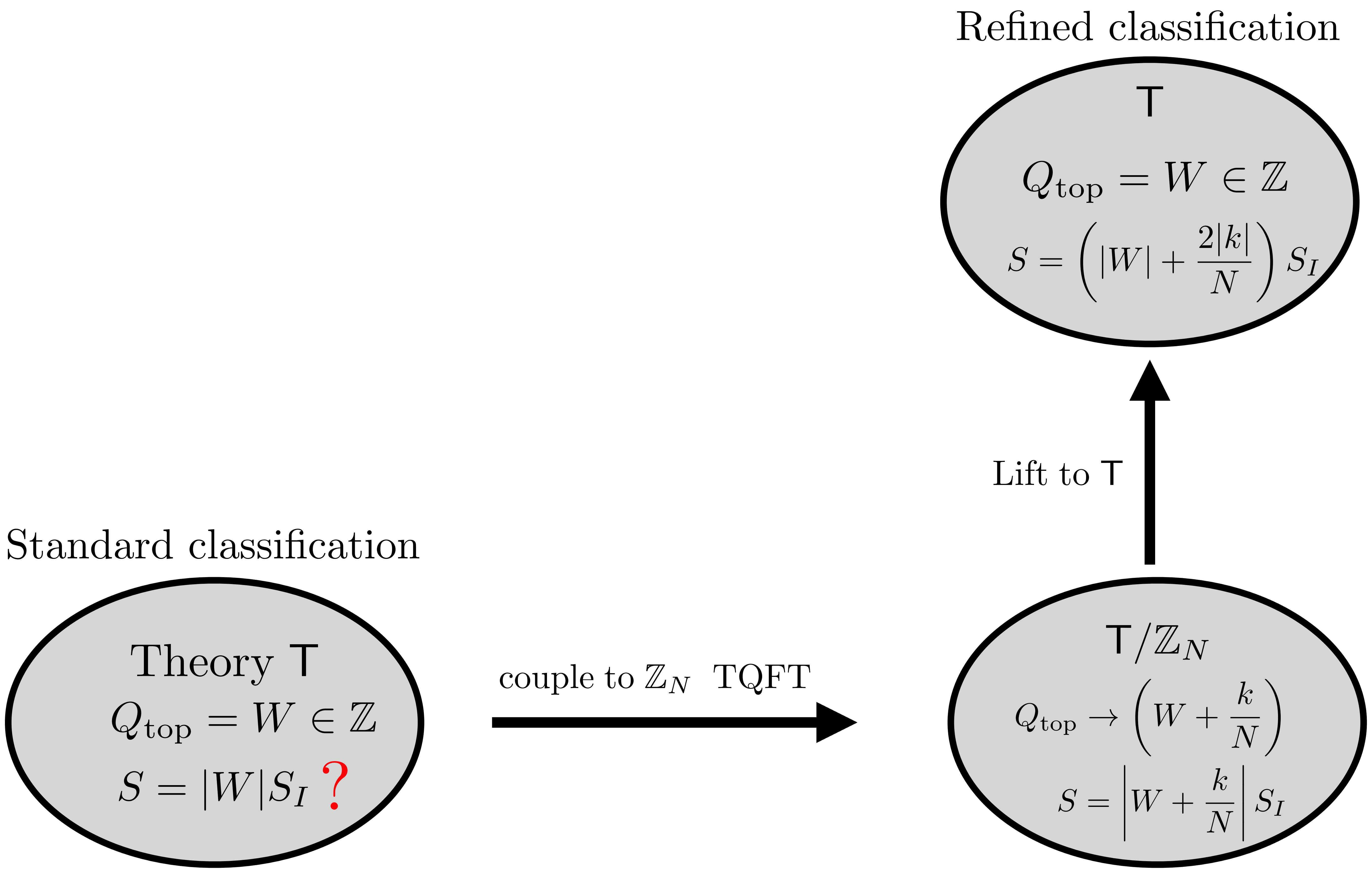
$$S = \frac{8\pi^2}{g^2}\left|W + \frac{(\ell\cdot m)}{N}\right| \in \frac{S_I}{N}\mathbb{Z}^{\geq 0}$$

Action of BPS saddles in PSU(N)
What is the implication for SU(N) and renormalon problem?

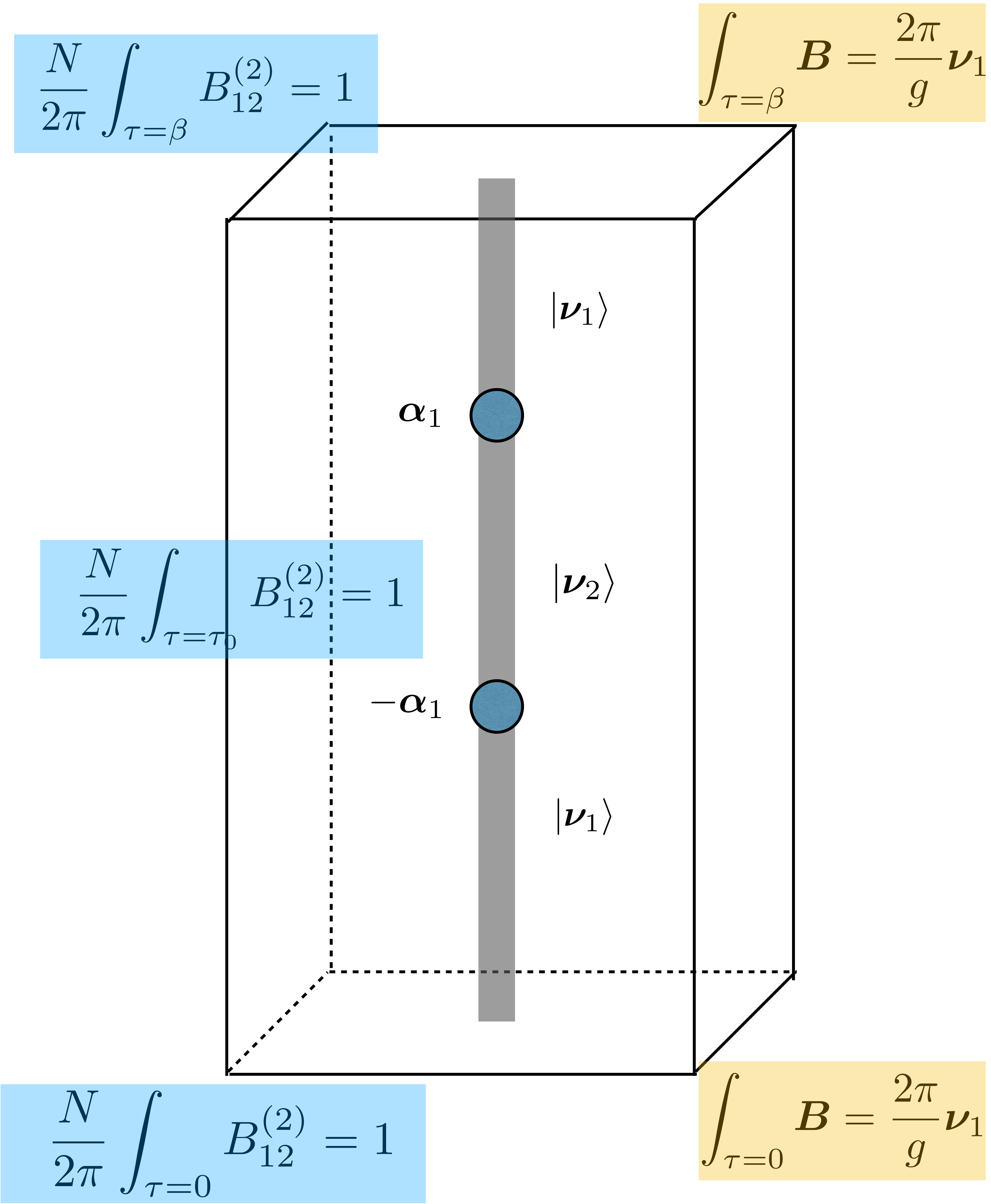
NP expansion on arbitrary size T_4 is controlled by S_I/N, but not S_I in SU(N) theory.

Fractional action, integer topological charge!

This is major difference from standard perspective.



Turn on 't Hooft flux background in 3-direction



Physical way to think about the 't Hooft flux:

A non-trivial background magnetic GNO flux in co-weight (modulo co-root) lattice corresponds to a non-zero 't Hooft flux. This is a non-dynamical center vortex. (See Greensite's review.)

Magnetic flux is easier to imagine than discrete flux. It is just Aharonov-Bohm flux for chromo-magnetic field. Magnetic N-ality of the magnetic flux is equal to 't Hooft flux:.

Gonzalez-Arroyo, Garcia Perez et.al. (1990s-): Important works around this idea: lattice simulations, on latticized T3 x R, that time-dependent fractional instanton solutions with action 1/N (no abelianization)

Classification of tunnelings in ’t Hooft flux background

But now, there is something more interesting. Consider the following magnetic flux configurations (all of which have the same ’t Hooft -flux), which can be connected by monopoles in root lattice.

$$\boldsymbol{\Phi} = \int_{T^2} \boldsymbol{B} = \frac{2\pi}{g}\boldsymbol{\nu}_a, \qquad a = 1, \ldots, N.$$

which are exactly degenerate.

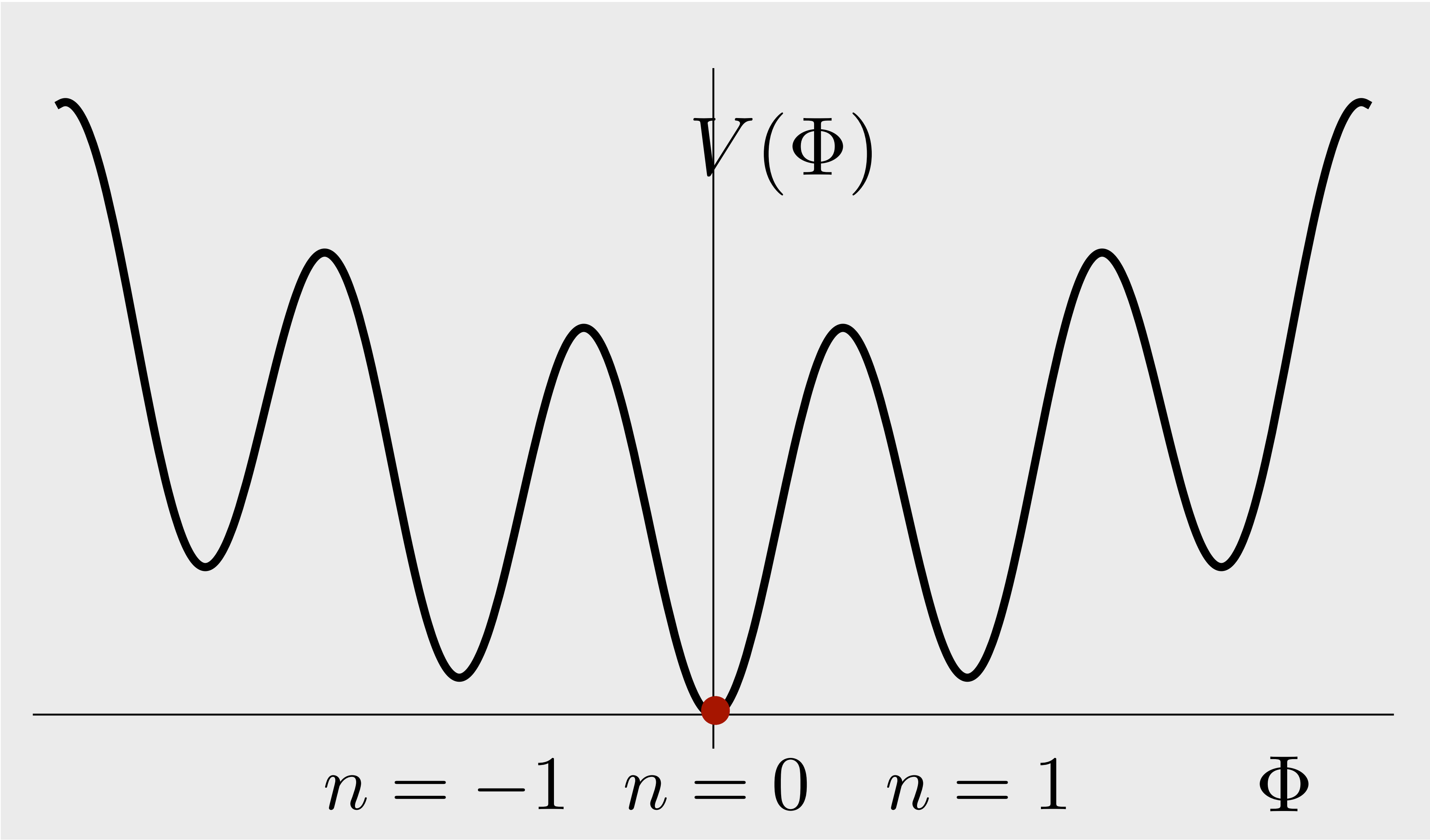
$$E_a = \tfrac{1}{2}\int_{T^2} {\boldsymbol{B}_a}^2 = \frac{1}{2A}\left(\frac{2\pi}{g}\right)^2 \boldsymbol{\nu}_a^2 = \frac{1}{2A}\left(\frac{2\pi}{g}\right)^2\left(1 - \frac{1}{N}\right), \qquad a = 1, \ldots, N. \qquad E_a - E_b = 0$$

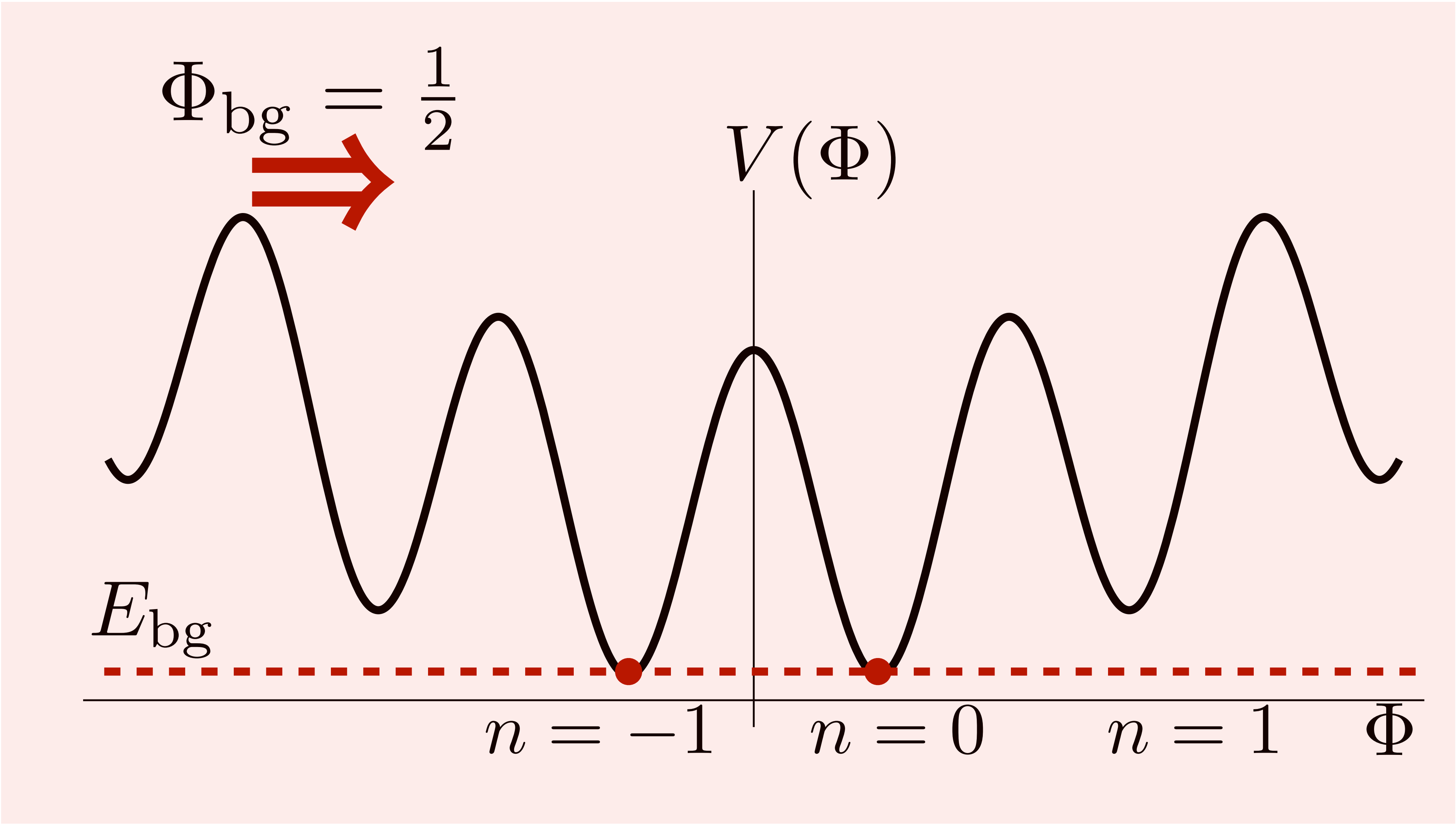
But the rest of all other magnetic flux configurations have higher energy at finite Area(T2) and become only degenerate in the infinite Area(T2) limit.

On finite T2 x R x S_L, there are **two types of tunnelings.**

1) Between states that becomes degenerate in Area(T2) tends to infinity limit. Eg. Polyakov, dYM both without TQFT background.
2) Between states that are already degenerate at finite Area(T2). This one is new, in the presence of TQFT background.

Reduction of SU(2) Polyakov model to QM with or without 't Hooft flux





On finite T2 x R, Polyakov model there are **two types of tunnelings.**

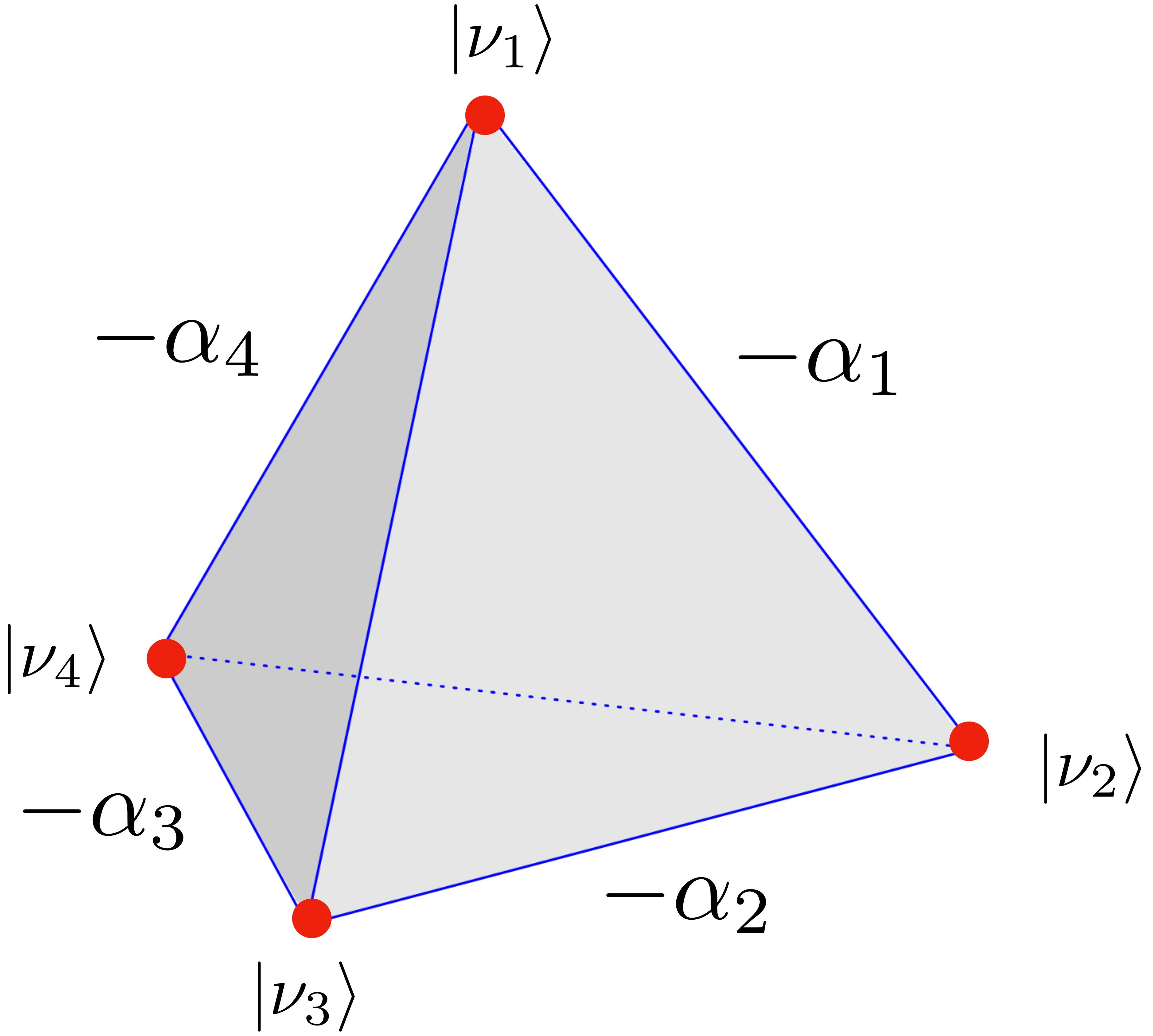
1) Between states that becomes degenerate in Area(T2) tends to infinity limit.

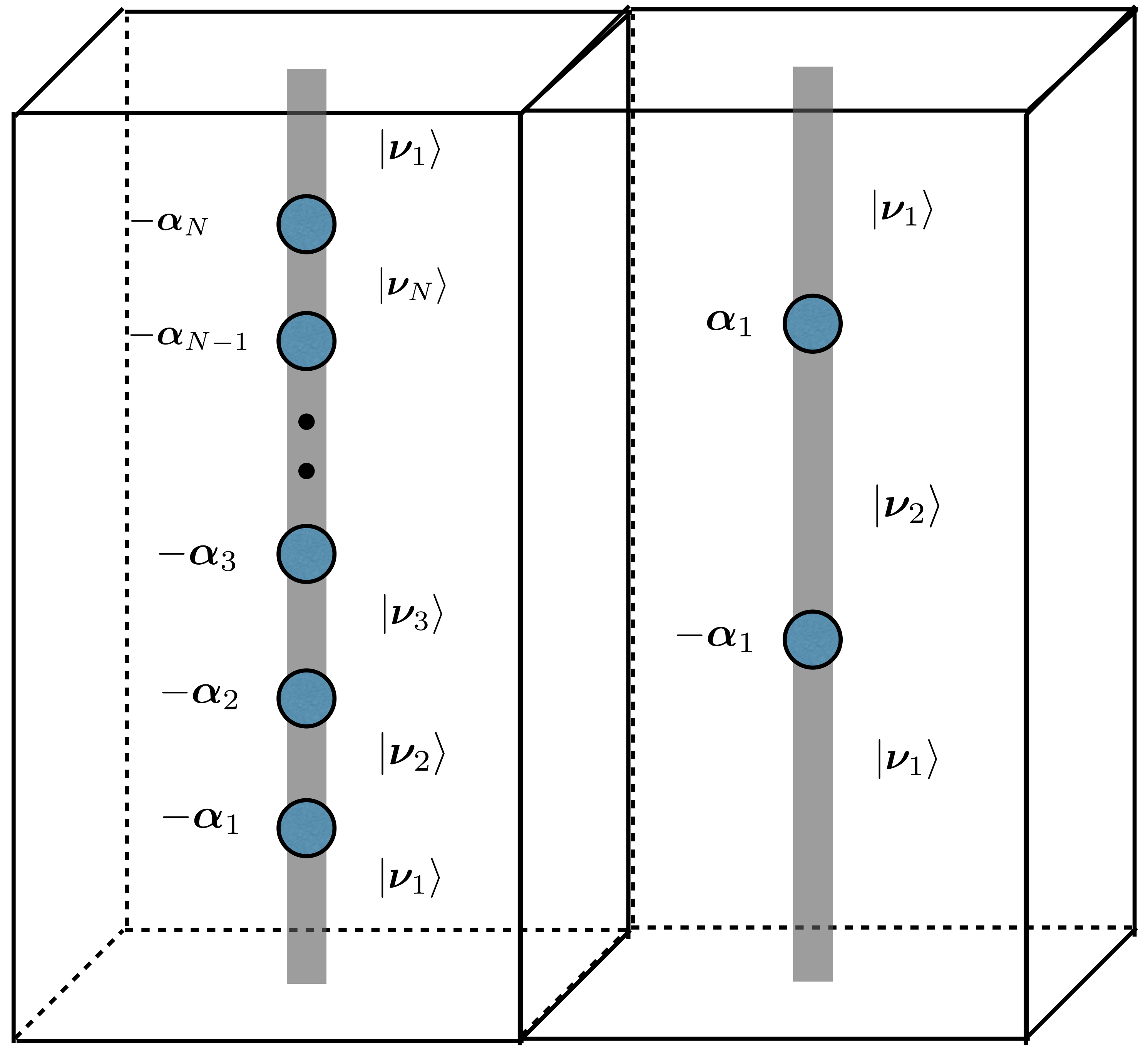
2) Between states that are already degenerate at finite Area(T2). This one is new, in the presence of TQFT background.

Born-Oppenheimer and particle on circle with N-min

In the small-T_2 limit, and within Born-Oppenheimer approximation, YM with center-symmetric holonomy along S_L reduces to quantum mechanical T_N model!

$$Z_{\ell_{12}} = \mathrm{tr}[\mathrm{e}^{-\beta H_{\ell_{12}}}] = \int_{\mathrm{pbc}} \mathcal{D}a \ \mathrm{e}^{-S(a,B_{12}^{(2)})}$$





There are N-induced classical minima due to classical Z_N background! In fact, this is one way of phrasing the origin of N-metastable vacua in YM theory!

Gluon condensate in semi-classical regime
(Either on R3 x S1 or QM on T3 x R)

$$\left\langle \frac{1}{N}\mathrm{tr}F^2 \right\rangle_{\pm}(\theta) = \max_q \Big[c_1 \Lambda^{\frac{11}{3}} \mathcal{R}^{-\frac{1}{3}} \cos\left(\frac{2\pi q + \theta}{N}\right)$$
$$+ c_2 \Lambda^{\frac{22}{3}} \mathcal{R}^{\frac{10}{3}} \cos 2\left(\frac{2\pi q + \theta}{N}\right) + \ldots \Big]$$
$$+ c_3 \Lambda^{\frac{22}{3}} \mathcal{R}^{\frac{10}{3}} \pm i\pi c_4 \Lambda^{\frac{22}{3}} \mathcal{R}^{\frac{10}{3}} + \ldots$$

$$t^* = \frac{1}{N} 2S_I = \frac{1}{N} t_{[I\bar{I}]}.$$

where $\mathcal{R} \equiv \mathcal{R}(LN)$ is the the fractional instanton size function, proportional to $LN \sim m_W^{-1}$ in the semi-classical regime.

Gluon condensate on large T4 or R4 is expected to be:

$$\left\langle \frac{1}{N}\mathrm{tr}F^2 \right\rangle_{\pm}(\theta) = \max_q \Lambda^4 h\left(\frac{\theta + 2\pi q}{N}\right) \pm i a \Lambda^4 ,$$

where $h(\theta/N)$ is $2\pi N$ periodic function (see Witten 1980)

Fourier decomposition = Lefschetz thimble decomposition

The appearance of $\mathrm{e}^{\mathrm{i}\frac{\theta}{N}k}, k \in \mathbb{Z}$ in condensate arises from calculable semi-classics, both on M_4 and $\mathbb{R}^3 \times S^1$. This is the weak coupling regime. The function that appears in $\mathbb{R}_4$ expression $h(\theta/N)$ can be expressed in terms of a complete Fourier basis:

$$\{\mathrm{Span}(\mathrm{e}^{\mathrm{i}\frac{\theta}{N}k}),\ k \in \mathbb{Z}\}\ .$$

Complete basis

The decomposition into $\mathrm{e}^{\mathrm{i}\frac{\theta}{N}k}, k \in \mathbb{Z}$ is inevitable, either on *small or large* M_4 as a consequence of thinking $SU(N)$ theory in TQFT backgrounds. All local non-perturbative observables must be multi-branched where each branch can be decomposed in the complete basis.

The completeness of the Fourier basis and its matching with the Lefschetz thimble decomposition of the partition function in terms of critical points gives us a realistic hope that the Lefschetz decomposition of path integral may well be complete even in QFT. (This is proven in QM examples by exact WKB).

Moving outside semi-classical domain with TQFT

Although semi-classics is not sufficient to address what happens as the theory moves from the semi-classical regime to the strong coupling regime, there is a sense in which it can be useful.

Polyakov argues that if the finite correlation length cuts off the growth of the fractional instanton size (as L gets larger in our case, recall that fractional instanton size is not a moduli) the EFT pushed to the boundary of its region of validity can provide a qualitatively accurate description.

$$\left\langle \frac{1}{N}\mathrm{tr}F^2 \right\rangle_{\pm}(\theta) = \max_q \Lambda^4 \left(\sum_{k\in\mathbb{Z}} c_k e^{i\frac{\theta+2\pi q}{N}k} \right) \pm ia\Lambda^4$$

$$t_{\text{bion}}^{\mathbb{R}^3\times S^1} = \frac{2S_I}{N} \implies t_{\text{ren.}}^{\mathbb{R}^4} = \frac{4S_I}{\beta_0}$$

Flow of Borel singularity, from semi-classical domain to strong coupling domain

The IR-renormalons in 4d gauge theories are composites of (semi-classical) fractional instantons configuration in the PSU(N) bundle that lifts to SU(N) bundle (neutral bions), and that are dressed by strong dynamics. The effect of dressing is to saturate the R-function around the strong scale, if the running coupling develops a singularity (pole, branch point) thereof.

As far as I can see, this is a unique way to make

1) theta angle periodicities and non-renormalization of theta
2) multi-branched structure of NP observables
3) weak coupling analysis on $R_3 \times S_1$
4) strong coupling expectation on R_4

simultaneously consistent.