Beyond Homogenization

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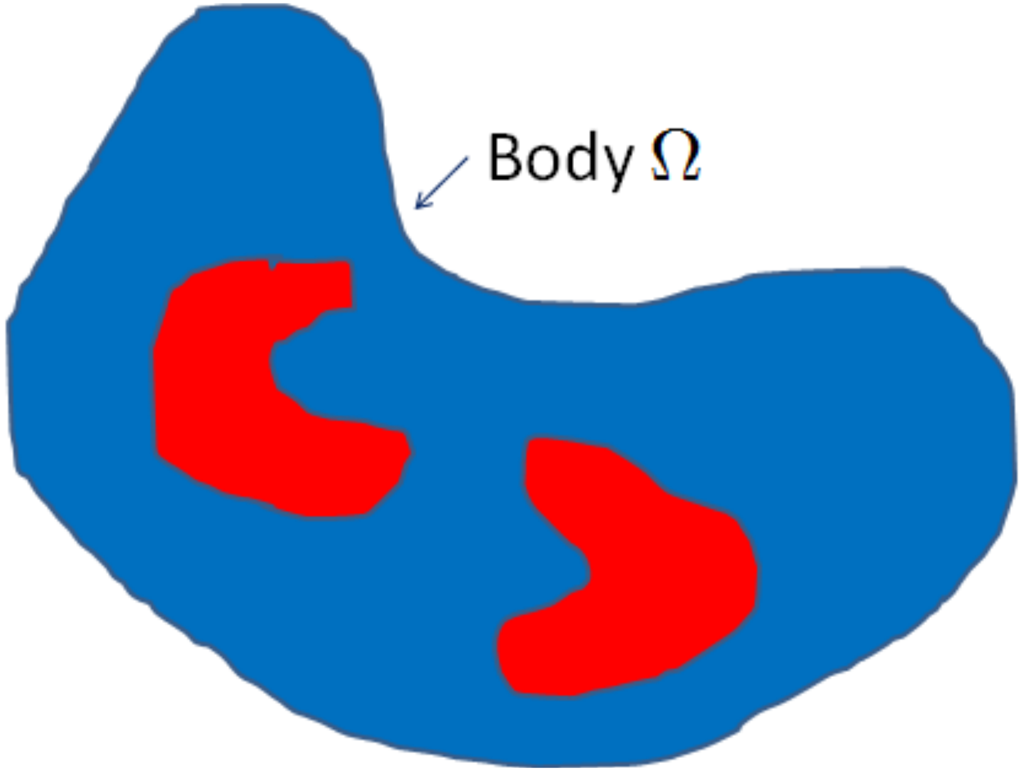
Model example: conductivity equations

$$\nabla \cdot \sigma \nabla V = 0, \quad \text{where} \quad \sigma(\mathbf{x}) = \chi(\mathbf{x})\sigma_1 + (1 - \chi(\mathbf{x}))\sigma_2,$$

Equivalently

$$\nabla \cdot \mathbf{j} = 0, \quad \mathbf{j} = \sigma \mathbf{e}, \quad \mathbf{e} = -\nabla V.$$



$$\langle \mathbf{e} \rangle = \frac{1}{|\Omega|} \int_{\partial\Omega} -V_0 \mathbf{n},$$

$$\langle \mathbf{j} \rangle = \frac{1}{|\Omega|} \int_{\partial\Omega} -\mathbf{x} q,$$

$$q(\mathbf{x}) = -\mathbf{n} \cdot \mathbf{j}(\mathbf{x})$$

Dirichlet Variational Principle: solution for V minimizes

$$W(\underline{V}) = \int_\Omega \sigma |\nabla \underline{V}|^2$$

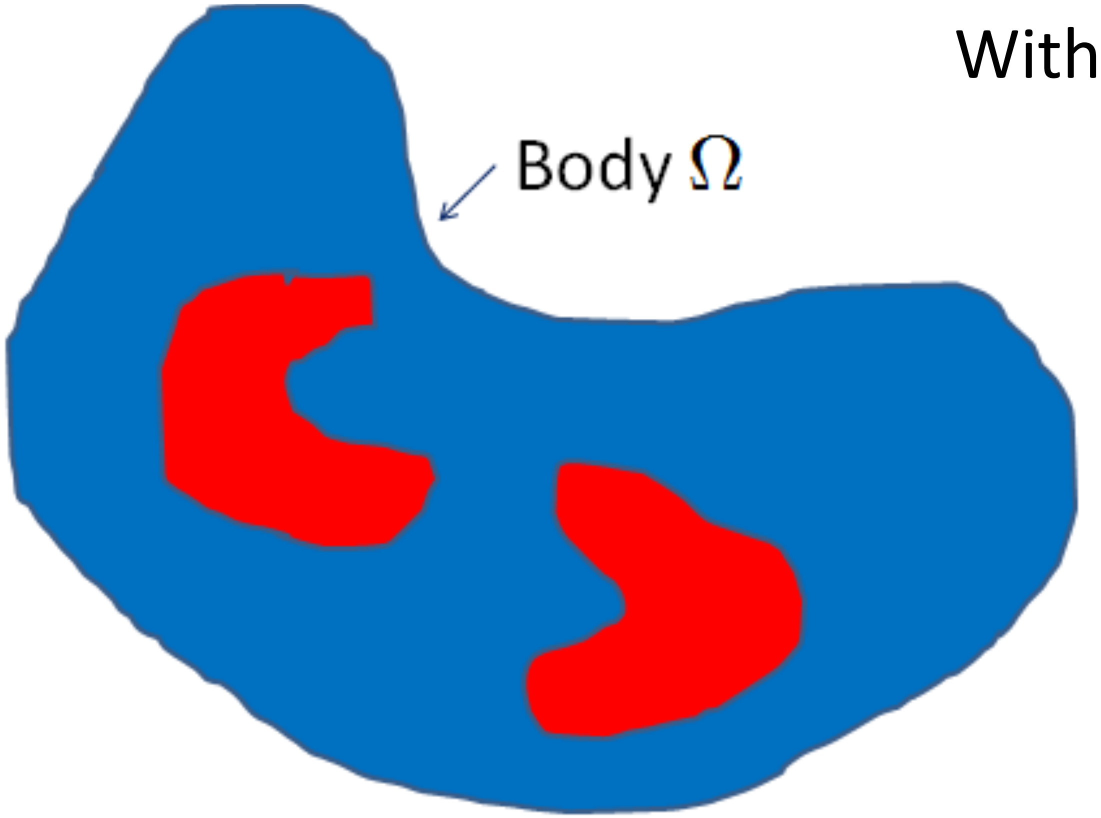
$$\text{with } \underline{V} = V \text{ on } \partial\Omega$$

Thompson Variational Principle: solution for $\mathbf{j}$ minimizes

$$\widetilde{W}(\underline{\mathbf{j}}) = \int_\Omega |\underline{\mathbf{j}}|^2/\sigma$$

$$\text{with } \underline{\mathbf{j}} \cdot \mathbf{n} = \mathbf{j} \cdot \mathbf{n} \text{ on } \partial\Omega \text{ and } \nabla \cdot \underline{\mathbf{j}} = 0 \text{ in } \Omega$$

The response of bodies with special boundary conditions:



With affine Dirichlet boundary conditions

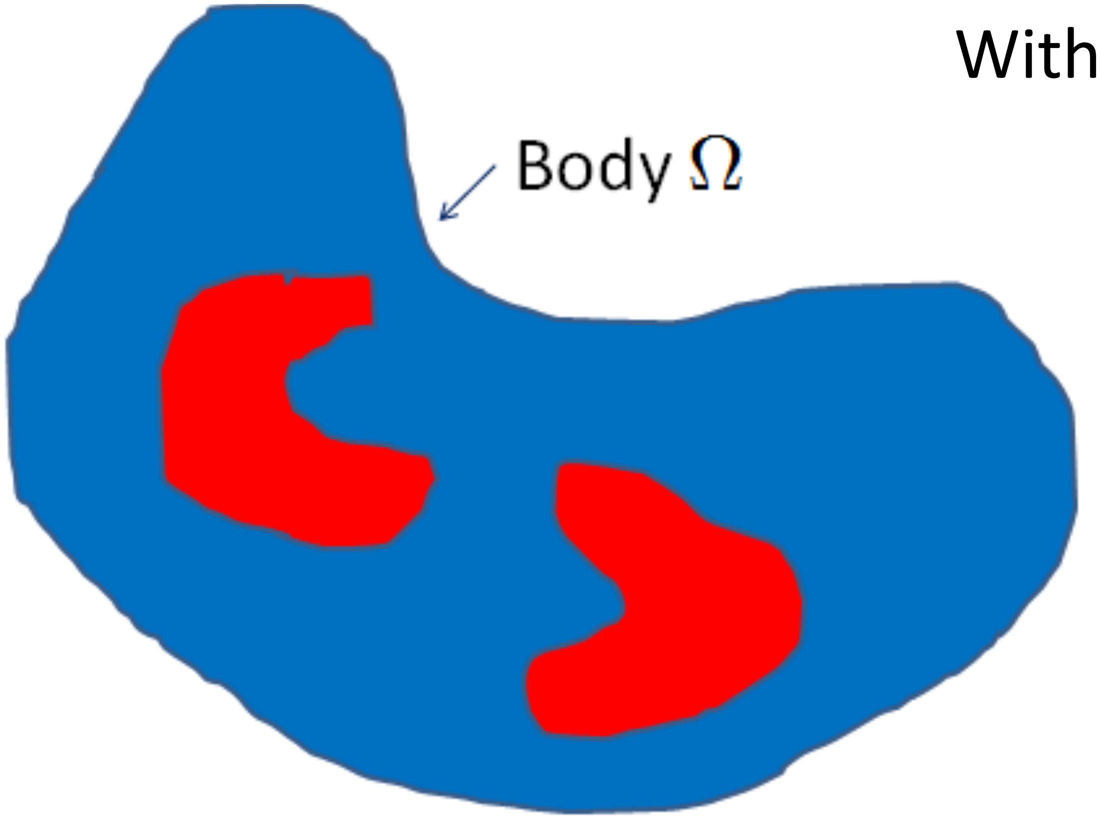
$$V_0 = -\mathbf{e}_0 \cdot \mathbf{x},$$

Measure $\mathbf{j}_0 = \langle \mathbf{j} \rangle.$

$$\mathbf{j}_0 = \boldsymbol{\sigma}^D \mathbf{e}_0,$$

Defines $\boldsymbol{\sigma}^D$

The response of bodies with special boundary conditions:



With special Neumann boundary conditions

$$q = \mathbf{j}_0 \cdot \mathbf{n},$$

Measure $\mathbf{e}_0 = \langle \mathbf{e} \rangle$

$$\mathbf{e}_0 = (\boldsymbol{\sigma}^N)^{-1}\mathbf{j}_0$$

Defines $\boldsymbol{\sigma}^N$

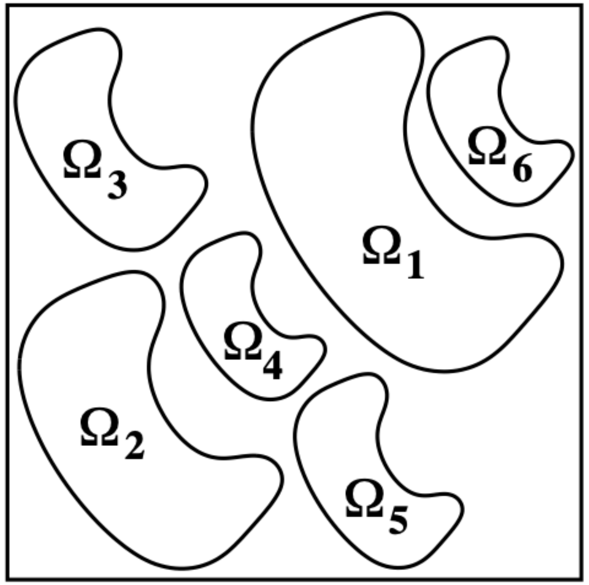
The standard Dirichlet-Thompson variational principles imply the bounds of Nemat-Nasser and Hori (1993):

$$\langle\sigma^{-1}\rangle^{-1}\mathbf{I} \leq \boldsymbol{\sigma}^D \leq \langle\sigma\rangle\mathbf{I}, \qquad \langle\sigma^{-1}\rangle^{-1}\mathbf{I} \leq \boldsymbol{\sigma}^N \leq \langle\sigma\rangle\mathbf{I}$$

In a two-phase body these can be used to estimate the volume fraction occupied by phase 1:

$$\left(\frac{\mathbf{e}_0\cdot\boldsymbol{\sigma}^D\mathbf{e}_0}{\mathbf{e}_0\cdot\mathbf{e}_0}-\sigma_2\right)/(\sigma_1-\sigma_2) \leq f_1 \leq \left(\sigma_2^{-1}-\frac{\mathbf{e}_0\cdot\mathbf{e}_0}{\mathbf{e}_0\cdot\boldsymbol{\sigma}^D\mathbf{e}_0}\right)/(\sigma_2^{-1}-\sigma_1^{-1})$$

$$\left(\frac{\mathbf{j}_0\cdot\mathbf{j}_0}{\mathbf{j}_0\cdot(\boldsymbol{\sigma}^N)^{-1}\mathbf{j}_0}-\sigma_2\right)/(\sigma_1-\sigma_2) \leq f_1 \leq \left(\sigma_2^{-1}-\frac{\mathbf{j}_0\cdot(\boldsymbol{\sigma}^N)^{-1}\mathbf{j}_0}{\mathbf{j}_0\cdot\mathbf{j}_0}\right)/(\sigma_2^{-1}-\sigma_1^{-1}).$$



For this periodic assemblage
the Dirichlet variational principle implies:

$$\boldsymbol{\sigma}^* \leq \boldsymbol{\sigma}^D,$$

while the Neumann variational principle implies:

$$\boldsymbol{\sigma}^* \geq \boldsymbol{\sigma}^N,$$

Murat-Tartar-Lurie-Cherkaev Bounds:

$$f_1 \mathrm{Tr}[(\boldsymbol{\sigma}^* - \sigma_2 \mathbf{I})^{-1}] \leq d/(\sigma_1 - \sigma_2) + f_2/\sigma_2, \quad f_2 \mathrm{Tr}[(\sigma_1 \mathbf{I} - \boldsymbol{\sigma}^*)^{-1}] \leq d/(\sigma_1 - \sigma_2) - f_1/\sigma_1,$$

imply:

$$f_1 \mathrm{Tr}[(\boldsymbol{\sigma}^D - \sigma_2 \mathbf{I})^{-1}] \leq d/(\sigma_1 - \sigma_2) + f_2/\sigma_2.$$

$$f_2 \mathrm{Tr}[(\sigma_1 \mathbf{I} - \boldsymbol{\sigma}^N)^{-1}] \leq d/(\sigma_1 - \sigma_2) - f_1/\sigma_1.$$

G.W.M. J.Mech.Phys.Sol. **60** 139-155, (2012)

Variational Principles for inhomogeneous bodies:

What is the extension of the Dirichlet and Thompson variational principles, as minimization principles

- To Acoustics?
- To Elastodynamics?
- To Electromagnetism?

The medium has to be lossy (absorbs energy) or the frequency ω has positive imaginary part (growing fields)

G.W.M, P. Seppecher, and G. Bouchitte, Proc. R. Soc. A.**465** 367–396 (2009)
G.W.M and J.R. Willis, Proc. R. Soc. A.**466** 3013–3032 (2010)

The quasistatic case was treated by Gibiansky and Cherkaev (1994).
Same as conductivity equations but all quantities complex.

$$\mathbf{D}(\mathbf{x}) = \varepsilon(\mathbf{x})\mathbf{E}(\mathbf{x}), \quad \nabla\cdot\mathbf{D} = 0, \quad \mathbf{E} = -\nabla V$$

$$\varepsilon''(\mathbf{x}) > 0$$

Separate into real and imaginary parts:

$$\begin{pmatrix} \mathbf{D}'' \\ \mathbf{D}' \end{pmatrix} = \begin{pmatrix} \varepsilon'' & \varepsilon' \\ \varepsilon' & -\varepsilon'' \end{pmatrix} \begin{pmatrix} \mathbf{E}' \\ \mathbf{E}'' \end{pmatrix}$$

Saddle shaped
quadratic form

Partial Legendre transforms convert saddle-shaped quadratic functions into convex quadratic functions.

Equivalent to rewriting constitutive law:

$$\begin{pmatrix} \mathbf{D}'' \\ \mathbf{E}'' \end{pmatrix} = \boldsymbol{\mathcal{E}} \begin{pmatrix} \mathbf{E}' \\ -\mathbf{D}' \end{pmatrix}$$

Lie in orthogonal subspaces!

$$\boldsymbol{\mathcal{E}} = \begin{pmatrix} \varepsilon'' + (\varepsilon')^2/(\varepsilon'') & \varepsilon'/\varepsilon'' \\ \varepsilon'/\varepsilon'' & 1/\varepsilon'' \end{pmatrix}$$

Positive Definite!

Cherkaev-Gibiansky minimization variational principles:

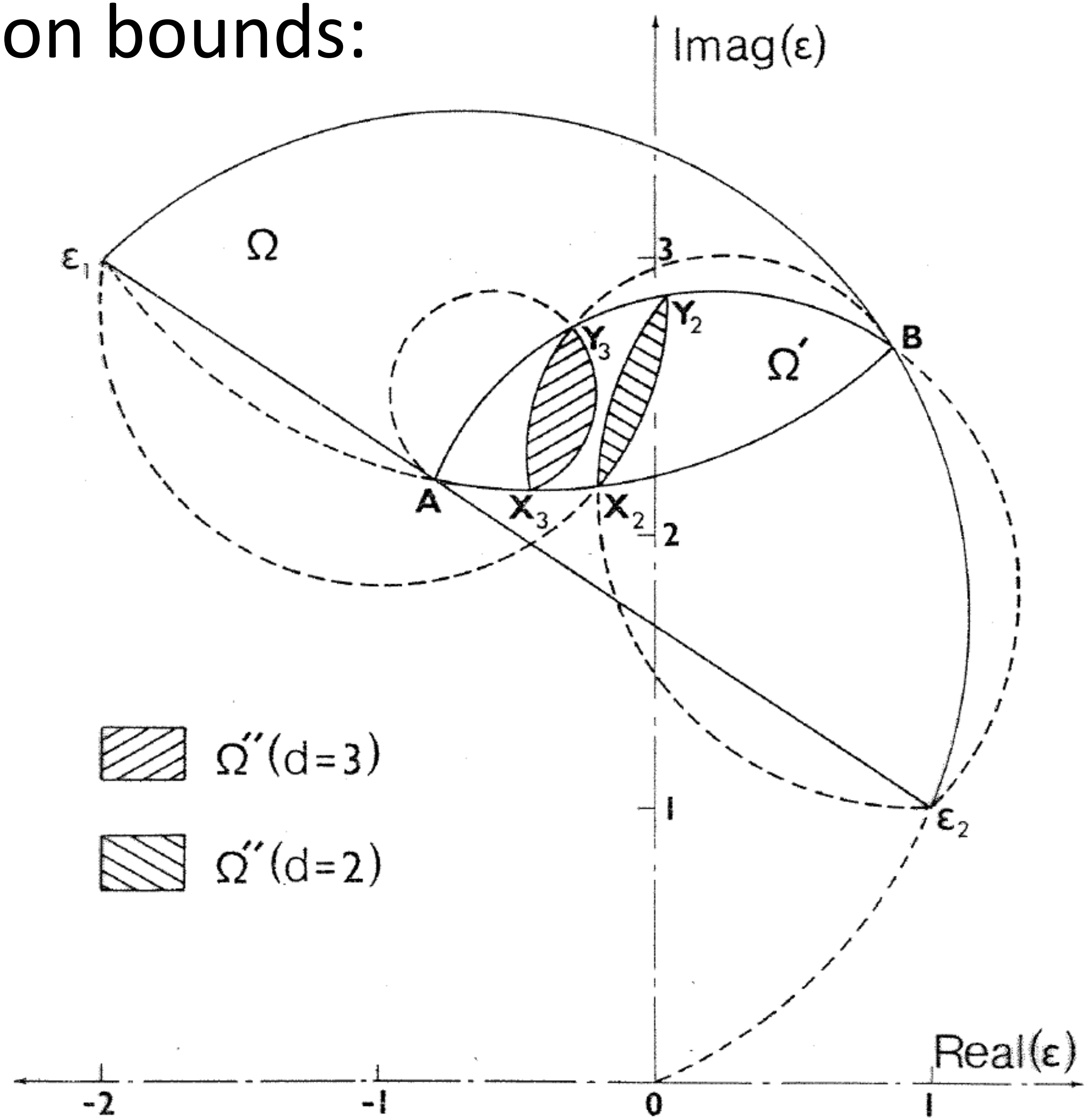
$$Y(V', \mathbf{D}') = \inf_{\underline{V}'} \quad \inf_{\substack{\underline{\mathbf{D}}' \\ \nabla \cdot \underline{\mathbf{D}}' = 0}} \quad Y(\underline{V}', \underline{\mathbf{D}}')$$

$$Y(\underline{V}', \underline{\mathbf{D}}') = \int_{\Omega} \begin{pmatrix} \nabla \underline{V}' \\ \underline{\mathbf{D}}' \end{pmatrix} \cdot \boldsymbol{\mathcal{E}} \begin{pmatrix} \nabla \underline{V}' \\ \underline{\mathbf{D}}' \end{pmatrix}$$

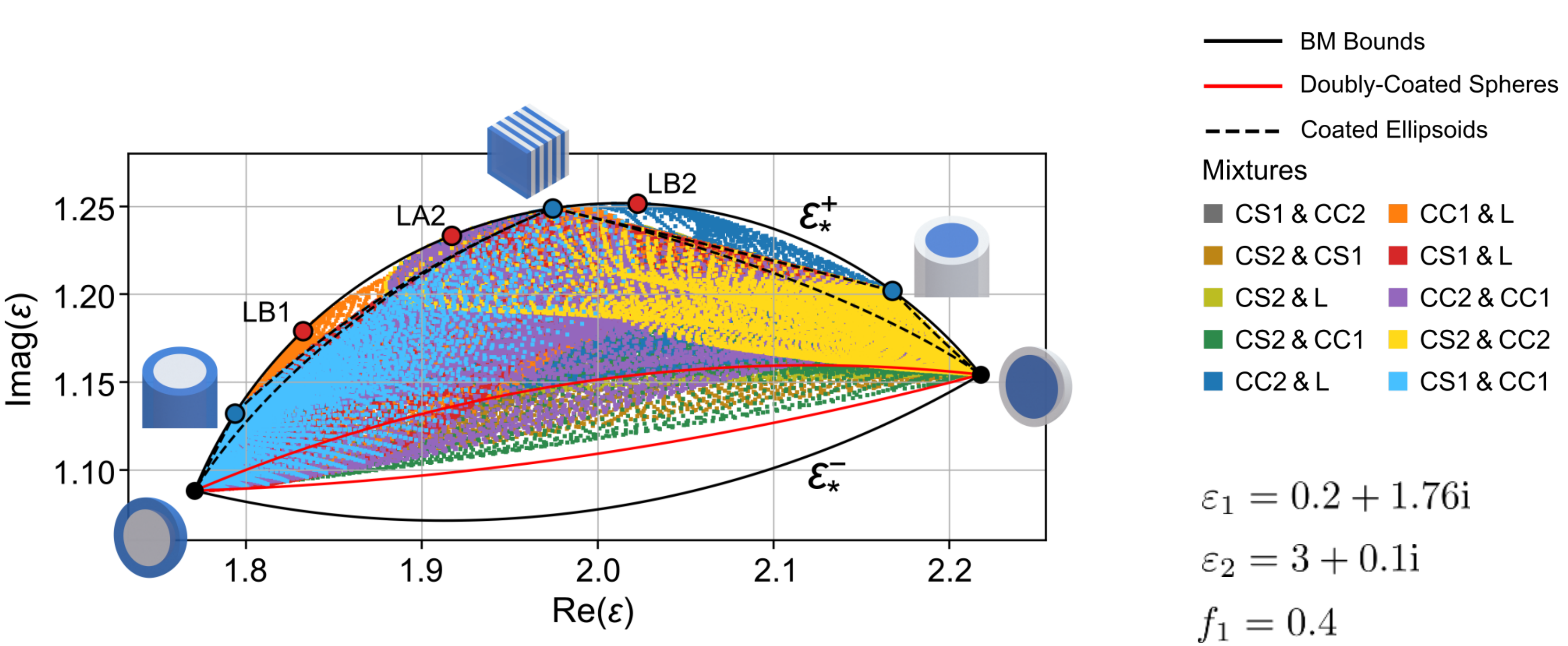
$$\underline{V}'(\mathbf{x}) = V'(\mathbf{x}), \qquad \underline{\mathbf{D}}' \cdot \mathbf{n} = \mathbf{D}' \cdot \mathbf{n} \quad \text{on} \quad \partial\Omega$$

Bergman-Milton bounds:

Based on the analytic properties of $\varepsilon_*(\varepsilon_1, \varepsilon_2)$



Recent work with C.Kern and O.D.Miller



$\varepsilon_1 = 0.2 + 1.76\mathrm{i}$

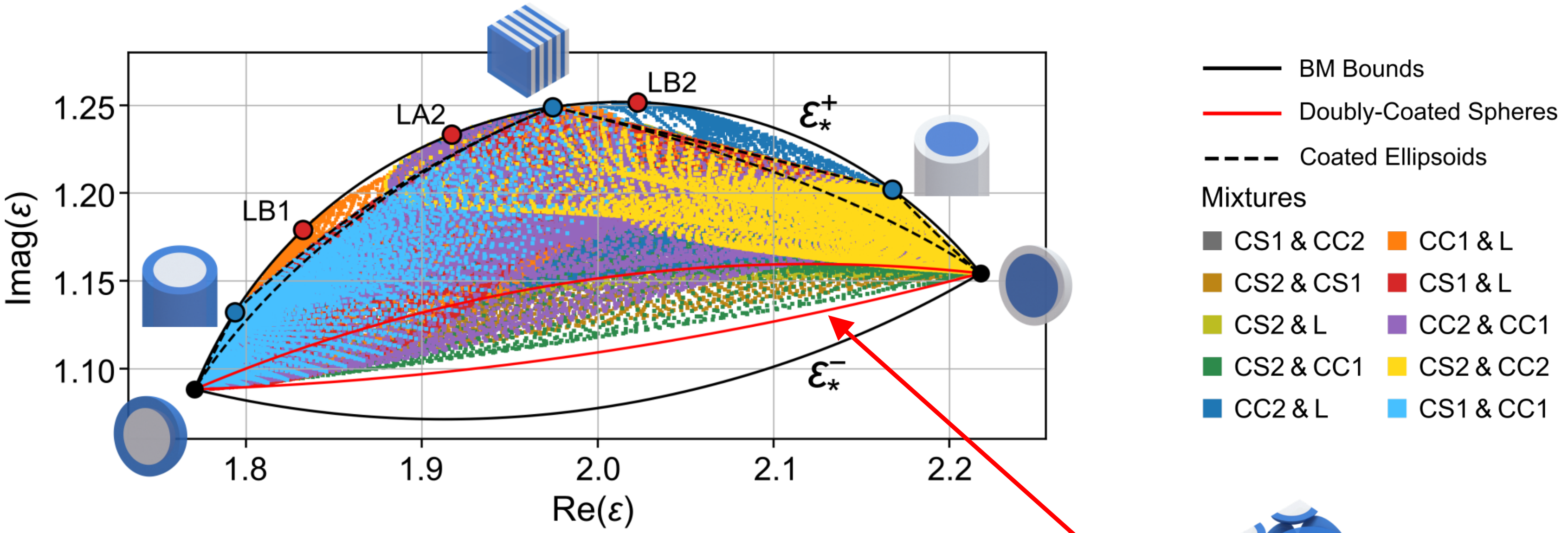
$\varepsilon_2 = 3 + 0.1\mathrm{i}$

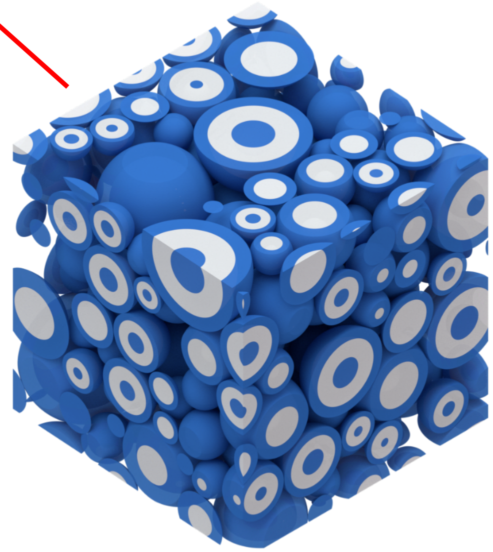
$f_1 = 0.4$

ε_*^+ seems to be at least almost optimal!

C. Kern, O. D. Miller, and G. W. Milton, Phys. Rev. Applied **14**, 054068 (2020)

Improved bound using the Cherkaev-Gibiansky transformation





What about the second arc, ε_*^-?

G.W.M (1980): "We suspect that Ω'' for d = 3 is not attainable […]"

G.W.M, Appl. Phys. Lett. **37**, 300 (1980)

C. Kern, O.D. Miller, and G.W.M., Phys. Rev. Applied **14**, 054068 (2020)

Key observation: the equations can be written in a similar form to the quasistatic dielectric ones:

Lie in orthogonal subspaces!

Acoustics:

$$\underbrace{\begin{pmatrix} -i\mathbf{v} \\ -i\nabla\cdot\mathbf{v} \end{pmatrix}}_{\mathcal{G}(\mathbf{x})} = \underbrace{\begin{pmatrix} -(\omega\rho)^{-1} & 0 \\ 0 & \omega/\kappa \end{pmatrix}}_{\mathbf{Z}(\mathbf{x})} \underbrace{\begin{pmatrix} \nabla P \\ P \end{pmatrix}}_{\mathcal{F}(\mathbf{x})},$$

Elastodynamics:

$$\underbrace{\begin{pmatrix} -\boldsymbol{\sigma}(\mathbf{x})/\omega \\ i\mathbf{p}(\mathbf{x}) \end{pmatrix}}_{\mathcal{G}(\mathbf{x})} = \underbrace{\begin{pmatrix} -\mathcal{C}/\omega & 0 \\ 0 & \omega\rho \end{pmatrix}}_{\mathbf{Z}(\mathbf{x})} \underbrace{\begin{pmatrix} \nabla\mathbf{u} \\ \mathbf{u} \end{pmatrix}}_{\mathcal{F}(\mathbf{x})},$$

Maxwell:

$$\underbrace{\begin{pmatrix} -i\mathbf{h} \\ i\nabla\times\mathbf{h} \end{pmatrix}}_{\mathcal{G}} = \underbrace{\begin{pmatrix} -[\omega\boldsymbol{\mu}(\mathbf{x})]^{-1} & 0 \\ 0 & \omega\boldsymbol{\varepsilon}(\mathbf{x}) \end{pmatrix}}_{\mathbf{Z}} \underbrace{\begin{pmatrix} \nabla\times\mathbf{e} \\ \mathbf{e} \end{pmatrix}}_{\mathcal{F}},$$

Schrödinger equation,

$$\begin{pmatrix} \mathbf{q}(\mathbf{x}) \\ \nabla\cdot\mathbf{q}(\mathbf{x}) \end{pmatrix} = \underbrace{\begin{pmatrix} -\mathbf{A} & 0 \\ 0 & E-V(\mathbf{x}) \end{pmatrix}}_{\mathbf{Z}(\mathbf{x})} \begin{pmatrix} \nabla\psi(\mathbf{x}) \\ \psi(\mathbf{x}) \end{pmatrix},$$

Variational principles for electromagnetism

Maxwell's equations:

$$\nabla \times \mathbf{E} = i\omega \mathbf{B}, \quad \nabla \times \mathbf{H} = -i\omega \mathbf{D} + \mathbf{j},$$

$$\mathbf{D} = \boldsymbol{\varepsilon} \mathbf{E}, \quad \mathbf{B} = \boldsymbol{\mu} \mathbf{H},$$

Minimization variational principle:
Define the positive definite matrices

$$\boldsymbol{\mathcal{E}} = \begin{pmatrix} \varepsilon'' + \varepsilon'(\varepsilon'')^{-1}\varepsilon' & \varepsilon'(\varepsilon'')^{-1} \\ (\varepsilon'')^{-1}\varepsilon' & (\varepsilon'')^{-1} \end{pmatrix},$$

$$\boldsymbol{\mathcal{M}} = \begin{pmatrix} \mathbf{m}'' + \mathbf{m}'(\mathbf{m}'')^{-1}\mathbf{m}' & \mathbf{m}'(\mathbf{m}'')^{-1} \\ (\mathbf{m}'')^{-1}\mathbf{m}' & (\mathbf{m}'')^{-1} \end{pmatrix}$$

$$\mathbf{m} = -\boldsymbol{\mu}^{-1}$$

Then: $$Y(\mathbf{E}', \mathbf{H}'') = \inf_{\underline{\mathbf{E}}'} \inf_{\underline{\mathbf{H}}''} Y(\underline{\mathbf{E}}', \underline{\mathbf{H}}''),$$

$$Y(\underline{\mathbf{E}}', \underline{\mathbf{H}}'') = \int_{\Omega} \begin{pmatrix} \omega\underline{\mathbf{E}}' \\ \nabla \times \underline{\mathbf{H}}'' - \mathbf{j}'' \end{pmatrix} \cdot \boldsymbol{\mathcal{E}} \begin{pmatrix} \omega\underline{\mathbf{E}}' \\ \nabla \times \underline{\mathbf{H}}'' - \mathbf{j}'' \end{pmatrix}$$

$$+ \begin{pmatrix} \nabla \times \underline{\mathbf{E}}' \\ -\omega\underline{\mathbf{H}}'' \end{pmatrix} \cdot \boldsymbol{\mathcal{M}} \begin{pmatrix} \nabla \times \underline{\mathbf{E}}' \\ -\omega\underline{\mathbf{H}}'' \end{pmatrix} + 2\omega\mathbf{j}' \cdot \underline{\mathbf{E}}',$$

$(\underline{\mathbf{E}}', \underline{\mathbf{H}}'')$ and $(\mathbf{E}', \mathbf{H}'')$ have the same tangential components at $\partial\Omega$

Other boundary conditions can be accommodated

When $\boldsymbol{\mu}$ is real : $Y(\mathbf{E}') = \inf_{\underline{\mathbf{E}}'} Y(\underline{\mathbf{E}}')$,

$$Y(\underline{\mathbf{E}}') = \int_\Omega 2\omega\mathbf{j}'\cdot\underline{\mathbf{E}}' + \begin{pmatrix} \omega\underline{\mathbf{E}}' \\ -\nabla\times\boldsymbol{\mu}^{-1}(\nabla\times\underline{\mathbf{E}}')/\omega - \mathbf{j}'' \end{pmatrix} \cdot \boldsymbol{\mathcal{E}} \begin{pmatrix} \omega\underline{\mathbf{E}}' \\ -\nabla\times\boldsymbol{\mu}^{-1}(\nabla\times\underline{\mathbf{E}}')/\omega - \mathbf{j}'' \end{pmatrix}$$

The infimum is over fields with prescribed tangential components of

$\underline{\mathbf{E}}'$ and $\boldsymbol{\mu}^{-1}\nabla\times\underline{\mathbf{E}}'$ at $\partial\Omega$

Can we understand things in a broader framework?

Abstract Theory of Composites

Hilbert Space $\mathcal{H} = \mathcal{U} \oplus \mathcal{E} \oplus \mathcal{J}$

Operator $\mathrm{L} : \mathcal{H} \to \mathcal{H}$

For all $\mathbf{E}_0 \in \mathcal{U}$

Solve $\mathbf{J}_0 + \mathbf{J} = \mathbf{L}(\mathbf{E}_0 + \mathbf{E})$

With $\mathbf{J}_0 \in \mathcal{U}, \quad \mathbf{J} \in \mathcal{J}, \quad \mathbf{E} \in \mathcal{E},$

Then $\mathbf{J}_0 = \mathbf{L}_* \mathbf{E}_0$ defines $\mathbf{L}_* : \mathcal{U} \to \mathcal{U}$

Example: Periodic Conducting Composites

$\mathcal{H}$ - Periodic fields that are square integrable over the unit cell

$\mathcal{U}$ - Constant vector fields

$\mathcal{E}$ - Gradients of periodic potentials

$\mathcal{J}$ - Fields with zero divergence and zero average value

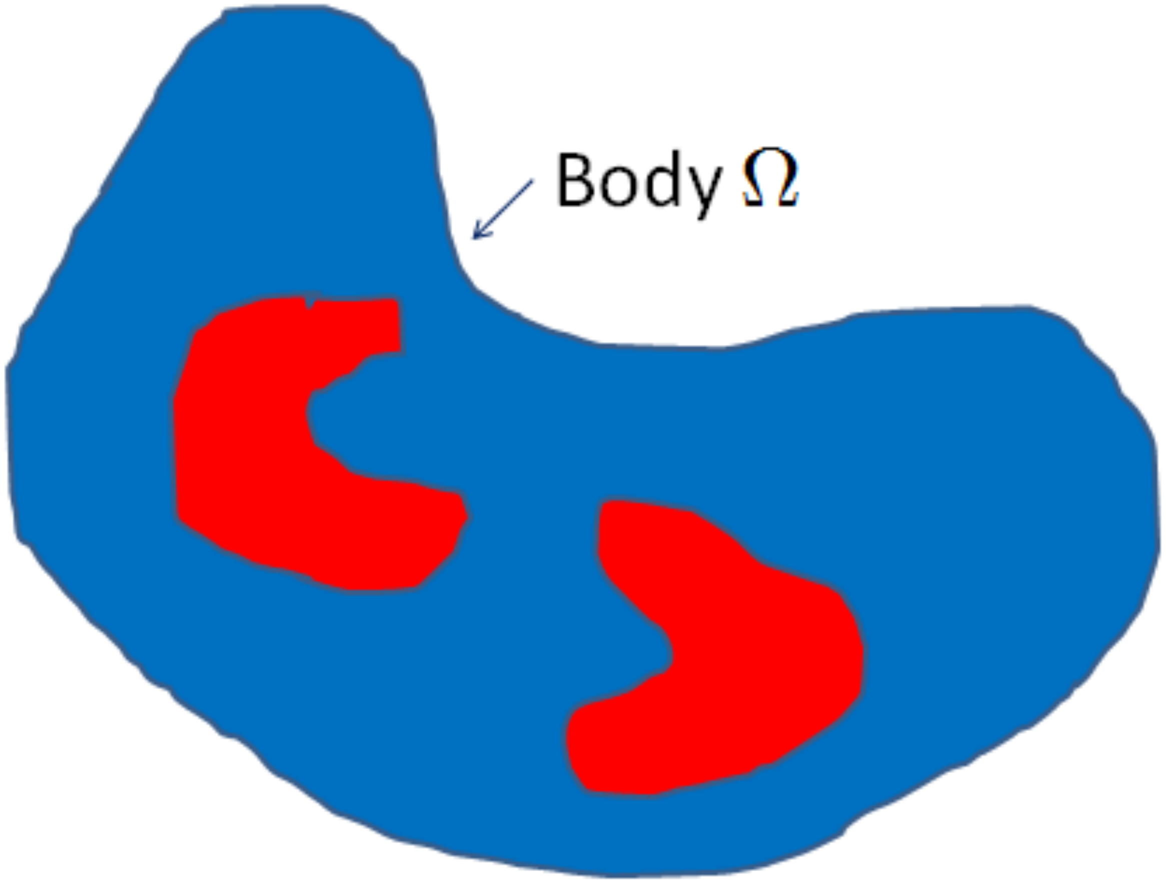
$\mathbf{E}_0 + \mathbf{E}(\mathbf{x})$ - Total electric field $\mathbf{e}(\mathbf{x})$

$\mathbf{J}_0 + \mathbf{J}(\mathbf{x})$ - Total current field $\mathbf{j}(\mathbf{x})$

$\mathrm{L} = \boldsymbol{\sigma}(\mathbf{x})$ - Local conductivity

$\mathbf{L}_* = \boldsymbol{\sigma}_*$ - Effective conductivity

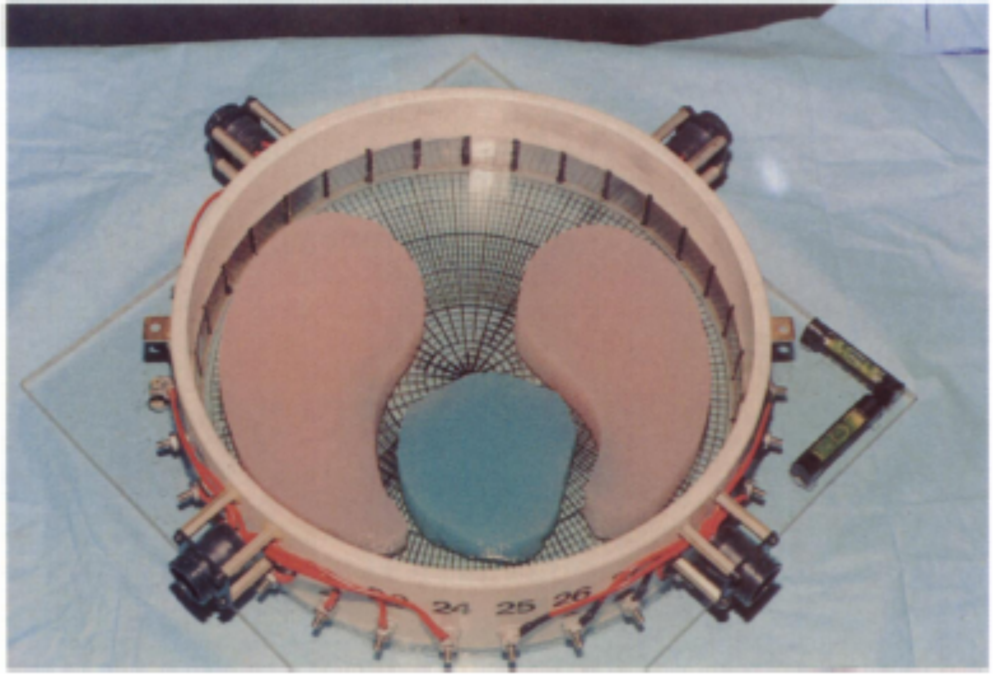
Dirichlet-to-Neumann Map

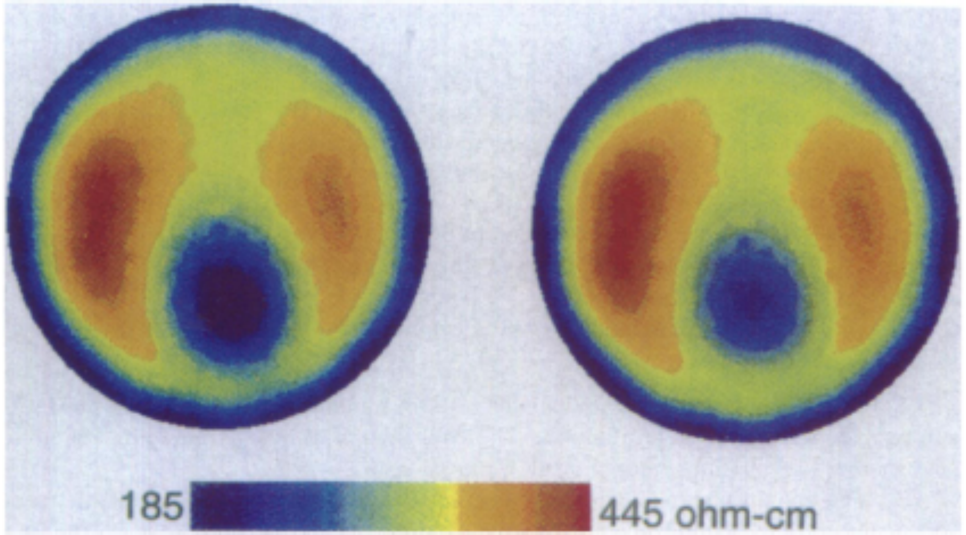


Specify boundary potential $V_0(\mathbf{x})$

Measure current flux $\mathbf{n} \cdot \mathbf{j}(\mathbf{x})$

Electrical Impedance Tomography: Cheney et. al. (1999)





The abstract theory of composites extends to this too (Grabovsky, 2016; Cassier, G.W.M, Welters, 2016)

Let

$\mathcal{U}$ consist of those fields $\mathbf{j} = \mathbf{e} = \nabla V$ where $V(\mathbf{x})$ is harmonic, $\nabla^2 V = 0$ inside Ω

$\mathcal{E}$ consist of fields $\mathbf{E} = -\nabla V$ with $V(\mathbf{x}) = 0$ on $\partial\Omega$

$\mathcal{J}$ consist of fields $\mathbf{J}$ with $\nabla \cdot \mathbf{J} = 0$ and $\mathbf{n} \cdot \mathbf{J} = 0$ on $\partial\Omega$

Three spaces are orthogonal

Note that those fields in $\mathcal{U}$ may either be identified by the boundary value of $V(\mathbf{x})$ or by the boundary value of the flux $\mathbf{n} \cdot \nabla V$

The abstract problem in composites consists in finding for a given field $\mathbf{E}_0(\mathbf{x})$ in $\mathcal{U}$ (with associated boundary potential $\mathrm{V}_0(\mathrm{x})$) the fields which solve:

$$\underbrace{\mathbf{J}_0(\mathbf{x}) + \mathbf{J}(\mathbf{x})}_{\mathbf{j}(\mathbf{x})} = \sigma(\mathbf{x}) \underbrace{[\mathbf{E}_0(\mathbf{x}) + \mathbf{E}(\mathbf{x})]}_{\mathbf{e}(\mathbf{x})}$$

with

$$\mathbf{J}_0 \in \mathcal{U},\ \mathbf{J} \in \mathcal{J},\ \text{and}\ \mathbf{E} \in \mathcal{E}$$

which is exactly the conductivity problem we would solve for the Dirichlet problem.

Furthermore if we knew the effective operator

$$\mathbf{L}_*: \mathcal{U} \to \mathcal{U}$$

Then we have

$$\mathbf{J}_0 = \mathbf{L}_* \mathbf{E}_0$$

and the boundary values of $\mathbf{J}_0 \cdot \mathbf{n} = \mathbf{j} \cdot \mathbf{n}$ allow us to determine the Dirichlet-to-Neumann map:

$\mathbf{L}_*$ is equivalent to the Dirichlet-to-Neumann map.

- Variational principles in the abstract theory of composites correspond to the Dirichlet and Thompson ones.
- Analytic properties extend to the DtN map

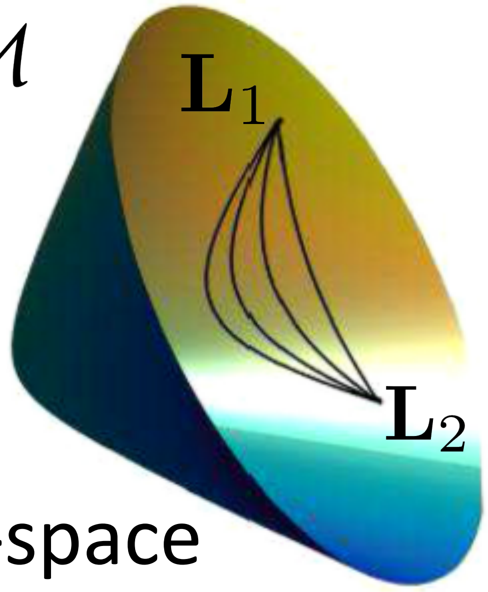
We formulated everything for conductivity but one may apply a similar procedure to many linear equations of physics

Extension of the theory of exact relations to inhomogeneous bodies

Classic example of an exact relation: Keller-Mendelson-Dykhne relation for 2-dimensional conductivity

$$\det \boldsymbol{\sigma}_* = c \quad \text{when } \det \boldsymbol{\sigma}(\mathbf{x}) = c \text{ for all } \mathbf{x}.$$

$\mathcal{M}$



$\mathbf{L}$ -space

Picture courtesy of Y. Grabovsky

The manifold
$\mathcal{M} = \{\boldsymbol{\sigma} : \det \boldsymbol{\sigma} = c\}$
is stable under homogenization.

Goal of the theory of exact relations:
identify manifolds of tensors, $\mathcal{M}$ that are
Stable under homogenization

Given periodic $\mathbf{L}(\mathbf{x})$ with $\mathbf{L}(\mathbf{x}) \in \mathcal{M}$ $\forall \mathbf{x}$ then $\mathbf{L}_* \in \mathcal{M}$

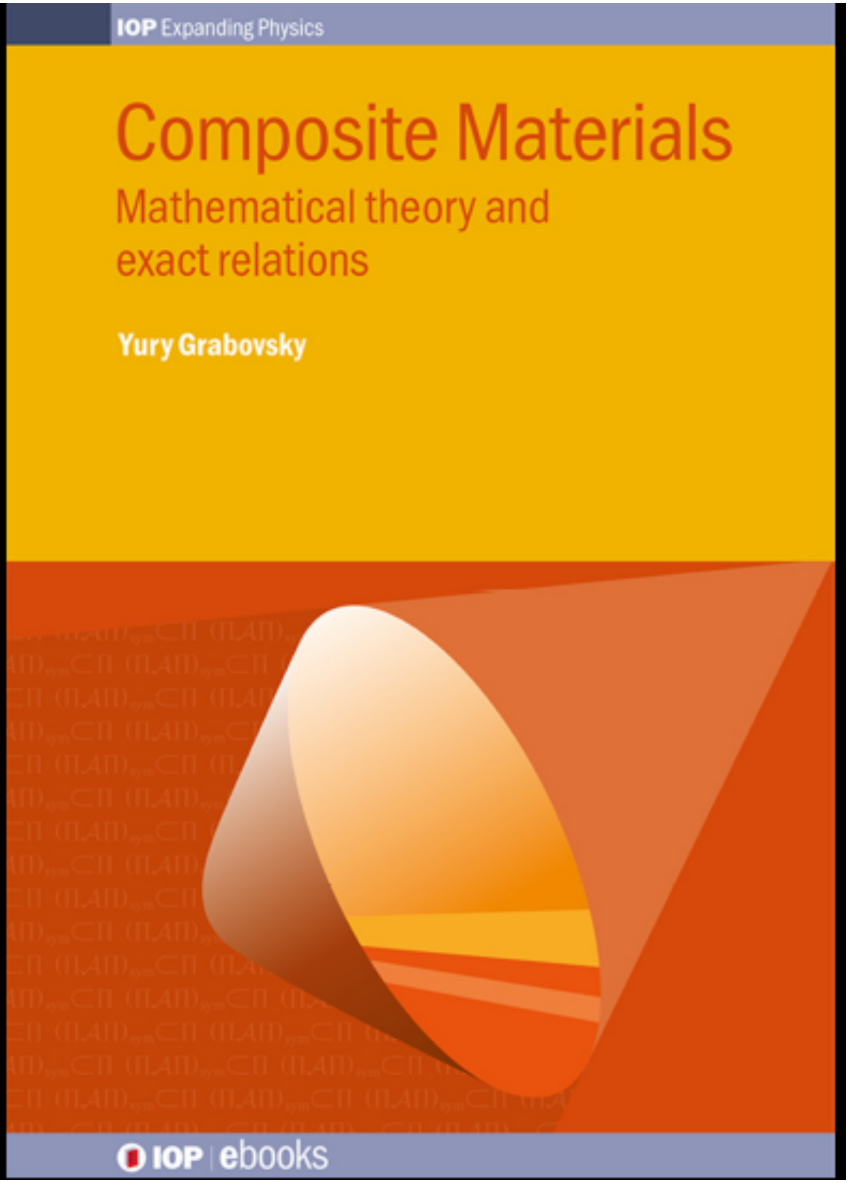
Many Scientists discovered exact relations one at a time:

Benveniste (Piezoelectricity)
Bergman (Hall-effect)
Berryman (Poroelasticity)
Chen (Coupled equations, Elasticity)
Cherkaev (Plate equations)
Cribb (Thermoelasticity)
Dvorak (Piezoelectricity)
Dykhne (Conductivity, Hall Effect)
Gassman (Poroelasticity)
Hashin (Elasticity)
He (Elasticity)
Helsing (Elasticity)
Hill (Elasticity)
Keller (Conductivity)

Levin (Thermoelasticity)
Lurie (Plate equations, Elasticity)
Matheron (Conductivity)
Milgrom (Coupled equations)
Milton (Complex conductivity, Hall effect, elasticity)
Movchan (elasticity)
Murat (Null-Lagrangians)
Shklovskii (Hall effect)
Shtrikman (Coupled equations)
Straley (Coupled Equations)
Strelniker (Hall effect)
Rosen (Thermoelasticity)
Schulgasser (Piezoelectricity)
Tartar (Null-Lagrangians)

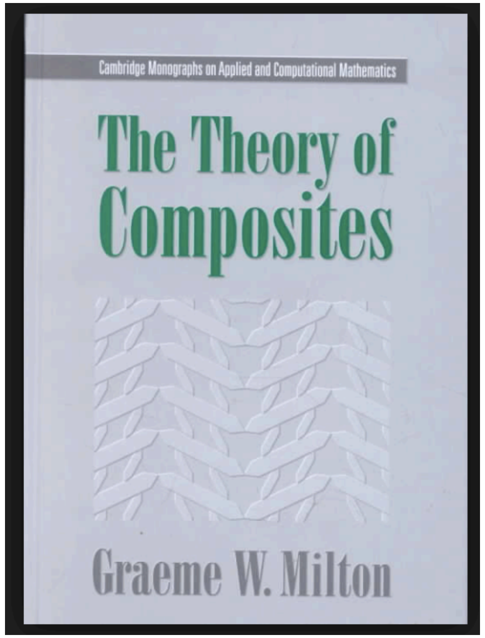
Yury Grabovsky and coworkers discovered hundreds,
(many intersections of more fundamental ones)

Theory of exact relations for composites reviewed in the books:



Grabovsky 2016

Milton 2002



Relevant Chapters:

3. **Duality transformations in two-dimensional media**
4. **Translations and equivalent media**
5. **Some microstructure-independent exact relations**
6. **Exact relations for coupled equations**
9. **Laminate materials**
12. **Reformulating the problem of finding effective tensors**
14. **Series expansions for the fields and effective tensors**
17. **The general theory of exact relations and links between effective tensors**

First major breakthrough: Grabovsky (1998)

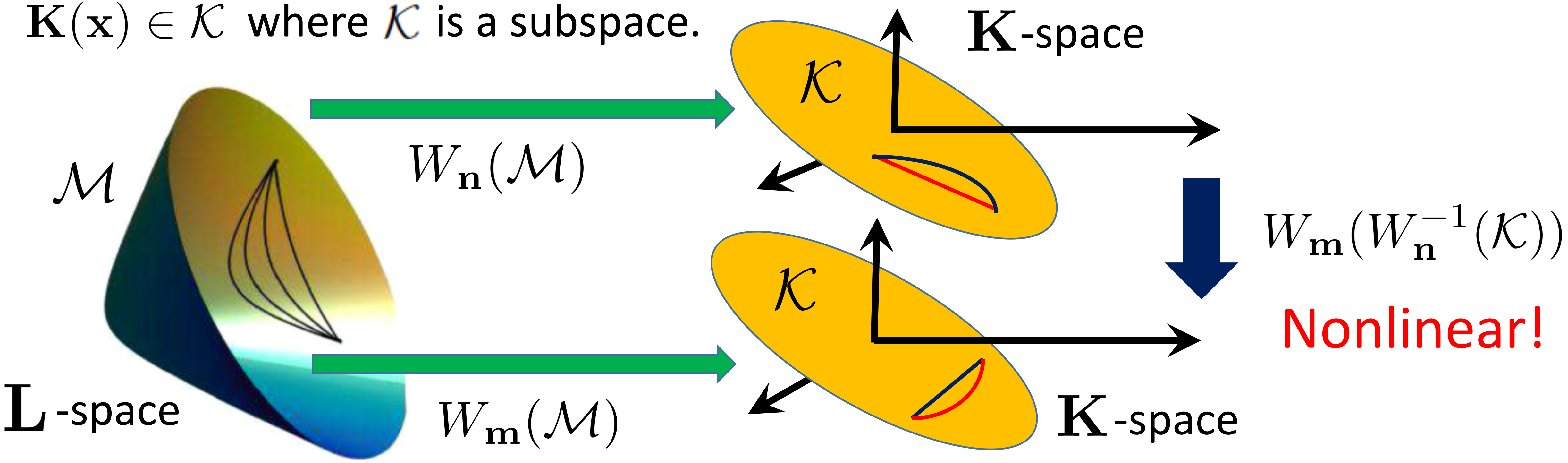
As an exact relation holds for all geometries it must certainly hold for laminate geometries

The transformation (G.W.M, 1990; Zhikov 1991)

$$W_{\boldsymbol{n}}(\boldsymbol{L}) = [\boldsymbol{I} + (\boldsymbol{L} - \boldsymbol{L}_0)\boldsymbol{\Gamma}(\boldsymbol{n})]^{-1}(\boldsymbol{L} - \boldsymbol{L}_0) = \boldsymbol{K}, \quad \mathbf{L}_0 \in \mathcal{M}$$

converts lamination in direction $\boldsymbol{n}$ to a linear average: $\mathbf{L}_* = W_{\mathbf{n}}^{-1}(\langle W_{\mathbf{n}}(\mathbf{L})\rangle)$

Therefore in $\boldsymbol{K}$ -space an exact relation must be a linear relation, $\mathbf{K}_* \in \mathcal{K}$, when $\mathbf{K}(\mathbf{x}) \in \mathcal{K}$ where $\mathcal{K}$ is a subspace.



Expansion of the non-linear transformation. Set $\boldsymbol{A}(\boldsymbol{m}) = \boldsymbol{\Gamma}(\boldsymbol{n}) - \boldsymbol{\Gamma}(\boldsymbol{m})$.

$$\begin{aligned} W_{\boldsymbol{m}}(W_{\boldsymbol{n}}^{-1}(\epsilon\boldsymbol{K})) &= \epsilon\boldsymbol{K}\{\boldsymbol{I} - [\boldsymbol{\Gamma}(\boldsymbol{n}) - \boldsymbol{\Gamma}(\boldsymbol{m})]\epsilon\boldsymbol{K}\}^{-1} \\ &= \epsilon\boldsymbol{K} + \epsilon^2\boldsymbol{K}\boldsymbol{A}(\boldsymbol{m})\boldsymbol{K} + \epsilon^3\boldsymbol{K}\boldsymbol{A}(\boldsymbol{m})\boldsymbol{K}\boldsymbol{A}(\boldsymbol{m})\boldsymbol{K} \\ &\quad + \epsilon^4\boldsymbol{K}\boldsymbol{A}(\boldsymbol{m})\boldsymbol{K}\boldsymbol{A}(\boldsymbol{m})\boldsymbol{K}\boldsymbol{A}(\boldsymbol{m})\boldsymbol{K} + \cdots, \end{aligned}$$

So $\mathcal{K}$ independent of $\boldsymbol{n}$ and

$\boldsymbol{K}\boldsymbol{A}(\boldsymbol{m})\boldsymbol{K} \in \mathcal{K}$ for all $\boldsymbol{m}$ and for all $\boldsymbol{K} \in \mathcal{K}$. (Necessary Condition)

Then all terms in the series lie in $\mathcal{K}$

The search for candidate exact relations becomes a search for subspaces $\mathcal{K}$ satisfying this algebraic constraint.

Example: Two-dimensional conductivity

Take $\mathbf{L}_0 = \sigma_0 \mathbf{I}$. Then

$$\mathbf{A}(\mathbf{m}) = \frac{\mathbf{n}\mathbf{n}^T}{(\sigma_0 \mathbf{n} \cdot \mathbf{n})} - \frac{\mathbf{m}\mathbf{m}^T}{(\sigma_0 \mathbf{m} \cdot \mathbf{m})}$$

is trace-free and symmetric. We can take $\mathcal{K}$ as the space of 2×2 symmetric trace-free matrices.

$$\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \begin{pmatrix} b & c \\ c & -b \end{pmatrix} = \begin{pmatrix} ab & ac \\ -ac & ab \end{pmatrix}$$

But with 3 matrices:

$$\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \begin{pmatrix} b & c \\ c & -b \end{pmatrix} \begin{pmatrix} d & e \\ e & -d \end{pmatrix} = \begin{pmatrix} abd + ace & abe - acd \\ abe - acd & -abd - ace \end{pmatrix}$$

Then $\mathcal{M} = W_{\mathbf{n}}^{-1}(\mathcal{K})$ consists of 2×2 symmetric matrices with determinant σ_0^2.

Second major breakthrough: (Grabovsky, G.W.M, Sage 2000)

The transformation $W_n(L)$ and series expansions of myself and Golden (1990) [that formed the basis of the rapidly converging FFT approach of Eyre and myself (1999)] provided the essential clues for a condition that guarantees a candidate exact relation holds for all geometries not just **laminate** ones.

$$\mathbf{K}(\mathbf{x}) = W_{\mathbf{M}}(\mathbf{L}(\mathbf{x})) = [\mathbf{I} + (\mathbf{L}(\mathbf{x}) - \mathbf{L}_0)\mathbf{M}]^{-1}(\mathbf{L}(\mathbf{x}) - \mathbf{L}_0)$$

$$\mathbf{K}_* = W_{\mathbf{M}}(\mathbf{L}_*) = [\mathbf{I} + (\mathbf{L}_* - \mathbf{L}_0)\mathbf{M}]^{-1}(\mathbf{L}_* - \mathbf{L}_0)$$

Series expansion: let $\boldsymbol{A}\boldsymbol{P} = \boldsymbol{M}(\boldsymbol{P} - \langle \boldsymbol{P} \rangle) - \boldsymbol{\Gamma}\boldsymbol{P}$ define $\boldsymbol{A}$ (acts locally in Fourier space)

$$\boldsymbol{K}_* = \langle [\boldsymbol{I} - \boldsymbol{K}\boldsymbol{A}]^{-1}\boldsymbol{K} \rangle = \sum_{j=0}^{\infty} \langle (\boldsymbol{K}\boldsymbol{A})^j \boldsymbol{K} \rangle \qquad \begin{aligned} \boldsymbol{A}(\boldsymbol{k}) &= \boldsymbol{M} - \boldsymbol{\Gamma}(\boldsymbol{k}) \quad \text{for } \boldsymbol{k} \neq 0, \\ &= 0 \qquad\qquad\quad \text{for } \boldsymbol{k} = 0. \end{aligned}$$

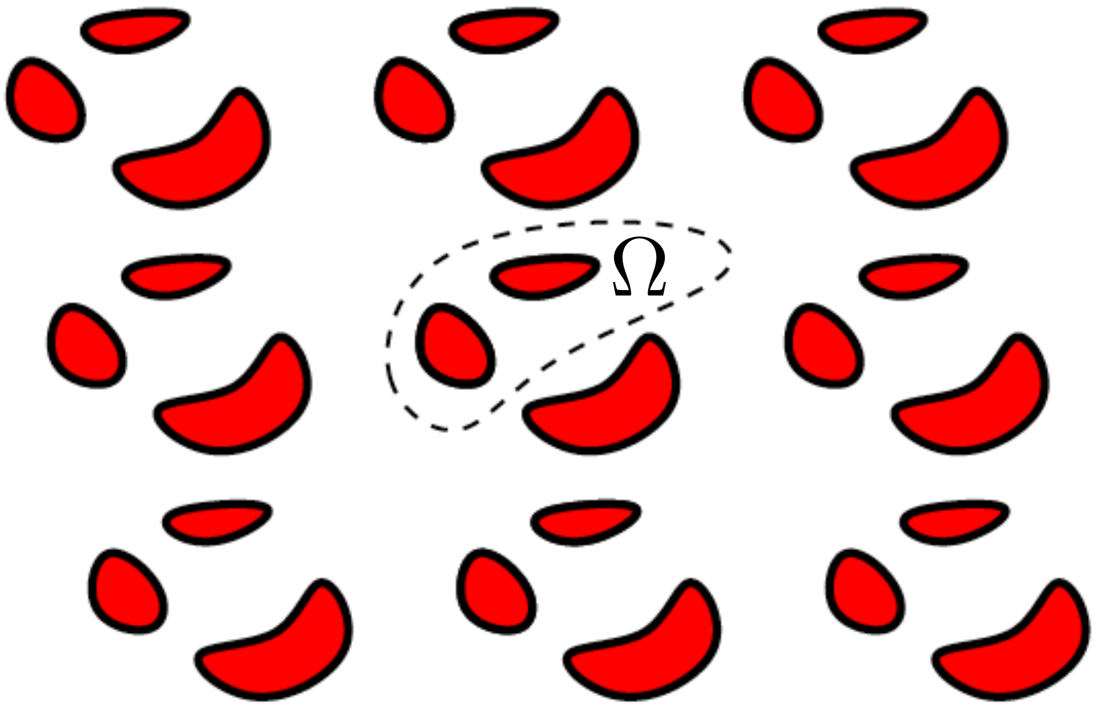
$\boldsymbol{K}_1[\boldsymbol{M} - \boldsymbol{\Gamma}(\boldsymbol{n})]\boldsymbol{K}_2 \in \overline{\mathcal{K}}$ and for all $\boldsymbol{K}_1, \boldsymbol{K}_2 \in \overline{\mathcal{K}}$, (Sufficient Condition)

Appropriately defined "polarization fields" within the material also are constrained to take values in $\overline{\mathcal{K}}$

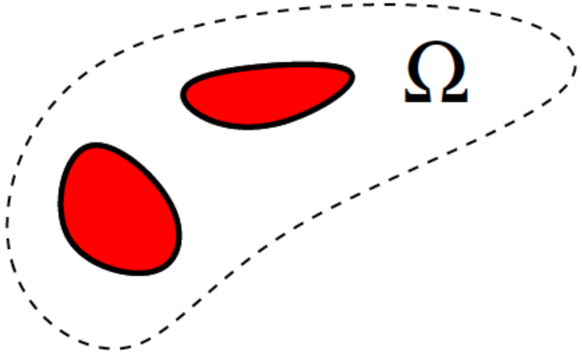
$\mathcal{K}$ can be taken to consist of all symmetric matrices in $\overline{\mathcal{K}}$

If the series does not converge, use analytic continuation

Third Major Breakthrough (G.W.M and Onofrei, Res Math Sci **6,** 19(2019))



Suppose we have a periodic composite for which an exact relation holds, And hence the "polarization field" takes values in $\overline{\mathcal{K}}$ at each $\boldsymbol{x}$ in Ω



The region Ω marked by the "dashed lines" does not know it is in a periodic medium, but the boundary conditions on the potentials or fluxes on this dashed boundary must be such to force the "polarization field" inside Ω to take values in $\overline{\mathcal{K}}$ and this gives us additional Information about the boundary fields.

Aim: identify these boundary conditions, and find the associated exact identities (**boundary field equalities**) satisfied by the "Dirichlet-to-Neumann map".

Key point:

These new boundary field equalities that in some sense generalize the divergence theorem, do not result from “integration by parts” but rather from algebraic properties tied with the operator Γ that is associated with the differential constraints satisfied by the fields on the left and right of the constitutive law.

There are “hidden identities” that go beyond integration by parts and still allow one to deduce exact identities satisfied by the fields at the boundary of a region Ω

Generalized viewpont of boundary field inequalities One eliminate $\mathbf{L}(\mathbf{x})$ from the the constitutive law $\mathbf{J}(\mathbf{x}) = \mathbf{L}(\mathbf{x})\mathbf{E}(\mathbf{x})$ and just view the constaint on $\mathbf{L}(\mathbf{x})$ that $\mathbf{L}(\mathbf{x}) \in \mathcal{M}$ as a constraint on the field pairs $(\mathbf{J}(\mathbf{x}), \mathbf{E}(\mathbf{x}))$ that is independent of $\mathbf{x}$.

For instance, if

- $\mathbf{E}$ consists of potential gradients,
- $\mathbf{J}$ consists of divergenge free fields (fluxes) that themselves may be expressed as curl's of additional potentials

Then collecting all potentials together as some grand potential $\mathbf{U}$,
The field constraints imply

$$\nabla \mathbf{U}(\mathbf{x}) \in \mathcal{A} \text{ for all } \mathbf{x} \in \Omega$$

where $\mathcal{A}$ is some non-linear manifold (determined by $\mathcal{M}$).

Then with appropriate nonlocal boundary conditions on the surface potential $\mathbf{U}(\mathbf{x}),\ \mathbf{x} \in \partial\Omega$ we obtain the constraint that

$$\nabla \mathbf{U}(\mathbf{x}) \in \mathcal{C} \text{ for all } \mathbf{x} \in \Omega$$

for some appropriately defined subspace $\mathcal{C}$, and this in turns constrains the tangential derivatives of $\mathbf{U}$ at $\partial\Omega$: these are the boundary field equalities.

Formulation

$$\sum_{i=1}^{d} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{d} \sum_{\beta=1}^{m} L_{i\alpha j\beta}(\mathbf{x}) \frac{\partial u_\beta(\mathbf{x})}{\partial x_j} \right) = f_\alpha(\mathbf{x}), \quad \alpha = 1, 2, \ldots, m,$$

Rewrite as

$$J_{i\alpha}(\mathbf{x}) = \sum_{j=1}^{d} \sum_{\beta=1}^{m} L_{i\alpha j\beta}(\mathbf{x}) E_{j\beta}(\mathbf{x}) - h_{i\alpha}(\mathbf{x}), \quad E_{j\beta}(\mathbf{x}) = \frac{\partial u_\beta(\mathbf{x})}{\partial x_j}, \quad \sum_{i=1}^{d} \frac{\partial J_{i\alpha}(\mathbf{x})}{\partial x_i} = 0,$$

with

$$\sum_{i=1}^{d} \frac{\partial h_{i\alpha}(\mathbf{x})}{\partial x_i} = f_\alpha(\mathbf{x})$$

Can extend the formulation to plate equations,
wave equations at constant frequency in lossy media, etc.

Exact relations for Green's Functions

$\mathbf{E}$ depends linearly on $\mathbf{h}$ and defines the (modified) infinite body Green's function in the inhomogeneous medium.

$$\mathbf{E}(\mathbf{x}) = \int_{\mathbb{R}^d} \mathbf{G}(\mathbf{x}, \mathbf{x}')\mathbf{h}(\mathbf{x}')\, d\mathbf{x}',$$

Define the "polarization field"

$$\mathbf{P}(\mathbf{x}) = \mathbf{J}(\mathbf{x}) - \mathbf{L}_0\mathbf{E}(\mathbf{x}) = [\mathbf{L}(\mathbf{x}) - \mathbf{L}_0]\mathbf{E}(\mathbf{x}) - \mathbf{h}(\mathbf{x})$$

Consider a point $\mathbf{x}^0$ and take $\mathbf{h}(\mathbf{x})$ to be proportional to a Dirac delta function localized at $\mathbf{x} = \mathbf{x}^0$:

$$\mathbf{h}(\mathbf{x}) = \mathbf{h}^0\delta(\mathbf{x} - \mathbf{x}^0), \quad \text{with } \mathbf{h}^0 = -(\mathbf{L}(\mathbf{x}^0) - \mathbf{L}_0)\mathbf{s}^0,$$

$$\mathbf{P}(\mathbf{x}) = (\mathbf{L}(\mathbf{x}^0) - \mathbf{L}_0)\mathbf{s}^0\delta(\mathbf{x} - \mathbf{x}^0) - (\mathbf{L}(\mathbf{x}) - \mathbf{L}_0)\mathbf{G}(\mathbf{x}, \mathbf{x}^0)(\mathbf{L}(\mathbf{x}^0) - \mathbf{L}_0)\mathbf{s}^0$$

Modified Green's function:

So $\mathbf{P}(\mathbf{x}) = \mathbf{T}(\mathbf{x}, \mathbf{x}_0)\mathbf{s}^0$ with

$$\mathbf{T}(\mathbf{x}, \mathbf{x}^0) = (\mathbf{L}(\mathbf{x}^0) - \mathbf{L}_0)\delta(\mathbf{x} - \mathbf{x}^0) - (\mathbf{L}(\mathbf{x}) - \mathbf{L}_0)\mathbf{G}(\mathbf{x}, \mathbf{x}^0)(\mathbf{L}(\mathbf{x}^0) - \mathbf{L}_0).$$

$$\mathbf{T} = (\mathbf{L} - \mathbf{L}_0) - (\mathbf{L} - \mathbf{L}_0)\mathbf{G}(\mathbf{L} - \mathbf{L}_0) = (\mathbf{I} - \mathbf{K}\boldsymbol{\Psi})^{-1}\mathbf{K},$$

As before

$$\mathbf{K}(\mathbf{x}) = W_{\mathbf{M}}(\mathbf{L}(\mathbf{x})) = [\mathbf{I} + (\mathbf{L}(\mathbf{x}) - \mathbf{L}_0)\mathbf{M}]^{-1}(\mathbf{L}(\mathbf{x}) - \mathbf{L}_0)$$

$$\boldsymbol{\Psi} = \mathbf{M} - \boldsymbol{\Gamma}$$

Expand:

$$\mathbf{T}(\mathbf{x},\mathbf{x}^0)=\delta(\mathbf{x}-\mathbf{x}^0)\mathbf{K}(\mathbf{x}^0)+\mathbf{K}(\mathbf{x})\widehat{\mathbf{\Psi}}(\mathbf{x}-\mathbf{x}^0)\mathbf{K}(\mathbf{x}^0)$$

$$+\int_{R^d}\mathbf{K}(\mathbf{x})\widehat{\mathbf{\Psi}}(\mathbf{x}-\mathbf{y}_1)\mathbf{K}(\mathbf{y}_1)\widehat{\mathbf{\Psi}}(\mathbf{y}_1-\mathbf{x}^0)\mathbf{K}(\mathbf{x}^0)\,d\mathbf{y}_1$$

$$+\int_{R^d}\int_{R^d}\mathbf{K}(\mathbf{x})\widehat{\mathbf{\Psi}}(\mathbf{x}-\mathbf{y}_1)\mathbf{K}(\mathbf{y}_1)\widehat{\mathbf{\Psi}}(\mathbf{y}_1-\mathbf{y}_2)\mathbf{K}(\mathbf{y}_2)\widehat{\mathbf{\Psi}}(\mathbf{y}_2-\mathbf{x}^0)\mathbf{K}(\mathbf{x}^0)\,d\mathbf{y}_1\,d\mathbf{y}_2+\ldots,$$

Upshot :

$\mathbf{T}(\mathbf{x},\mathbf{x}_0)$ takes values in $\overline{\mathcal{K}}$ when $\mathbf{L}(\mathbf{x})$ takes values in $\mathcal{M}$.

In the same way that one gets links between effective tensors so too can one get links between Green's functions of different physical problems (in inhomogeneous media)

I did not discuss how to get the “boundary field equalities” satisfied by the “Dirichlet to Neumann Map”.

The basic idea here (following Thaler and myself, 2014, where for a body Ω containing 2-phases sharing the same shear modulus, the boundary field equalities give the volume fraction occupied by one phase in the body) is to choose nonlocal boundary conditions that mimic the body Ω embedded in an infinite medium with appropriate sources outside that ensure the appropriately defined polarization field takes values in the subspace $\overline{\mathcal{K}}$

Also, generally, to reveal the exact relations satisfied by the DtN map one applies not just one boundary condition but a succession of them.

For more details see : G.W.M and Onofrei Res. Math Sci. **6,** 19 (2019)

Thank you!

Thank you!

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