Stable envelopes: a very brief introduction.

Basic idea:

$(\mathbb{C}^\times)^n = A \curvearrowright X, \omega$

Goal: construct $H_{eq}(X^A) \longrightarrow H_{eq}(X)$

Note: restriction gives $H_{eq}(X) \longrightarrow H_{eq}(X^A)$

an isomorphism after inverting some

$a \in H_{eq}(pt)$.

Stable envelopes give a different isomorphism.

K-theoretic stable envelopes [MO]

let $\mathcal{E} \in K_A(X)$, $\quad p \in X^A$

isolated fixed point.

$\mathcal{E}|_p \in \mathrm{Rep}(A)$.

$$\mathcal{E}|_p = \sum_{\mu \in X^\bullet(A)} c_\mu t^\mu$$

$\deg_A(\mathcal{E}|_p) =$ convex hull of the μ

$\subset \mathrm{Lie}(A)^\bullet_{\mathbb{R}}$

We can order degrees by containment:

$$\deg_A(\mathcal{E}|_p) \leq \deg_A(\mathcal{E}'|_p)$$

if one polytope is contained in the other.

$A \curvearrowright X, \omega$ hamiltonian torus action

$\cap$

$T \curvearrowright X$ non-hamiltonian torus action

(scales ω by a character $\kappa : T \to \mathbb{C}^\times$.)

To define K-theoretic stable envelopes,

we fix some aux. data:

1) Cocharacter $\sigma: \mathbb{C}^\times \longrightarrow A$

s.t. $X^{\mathbb{C}^\times} = X^A$.

2) Polarization

$$TX = T^{1/2} + \hbar^{-1}(T^{1/2})^\vee$$
$$\in K_T(X)$$

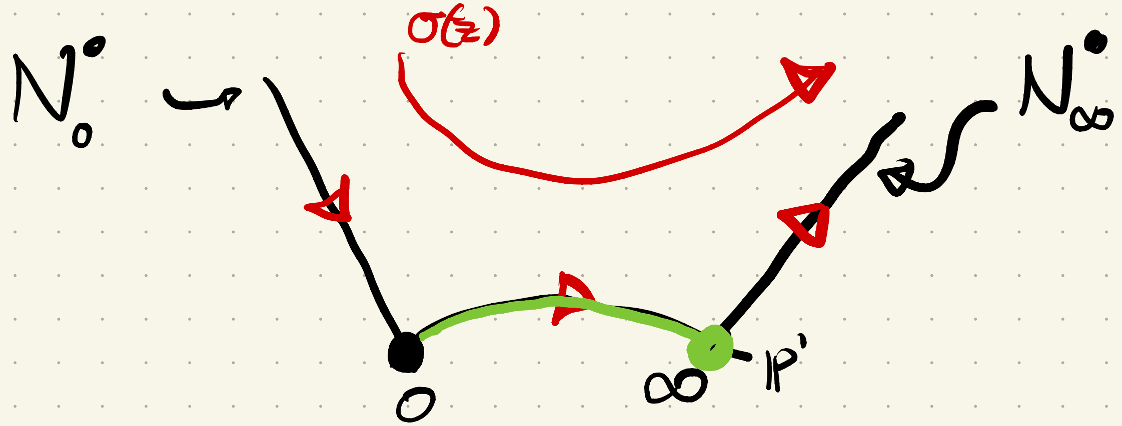
3) Slope $\mathcal{L} \in \mathrm{Pic}_A(X) \otimes_{\mathbb{Z}} \mathbb{Q}$

s.t. $\deg(\mathcal{L}|_{C_{pq}}) \notin \mathbb{Z}$ where C_{pq} is any rat. curve between $p, q \in X^A$.

Given $p \in X^A$, define

$$\mathrm{Attr}_\sigma(p) = \{ x \in X \mid \lim_{z \to 0} \sigma(z)x = p \}.$$

<u>Example:</u> $X = T^*\mathbb{P}^1$

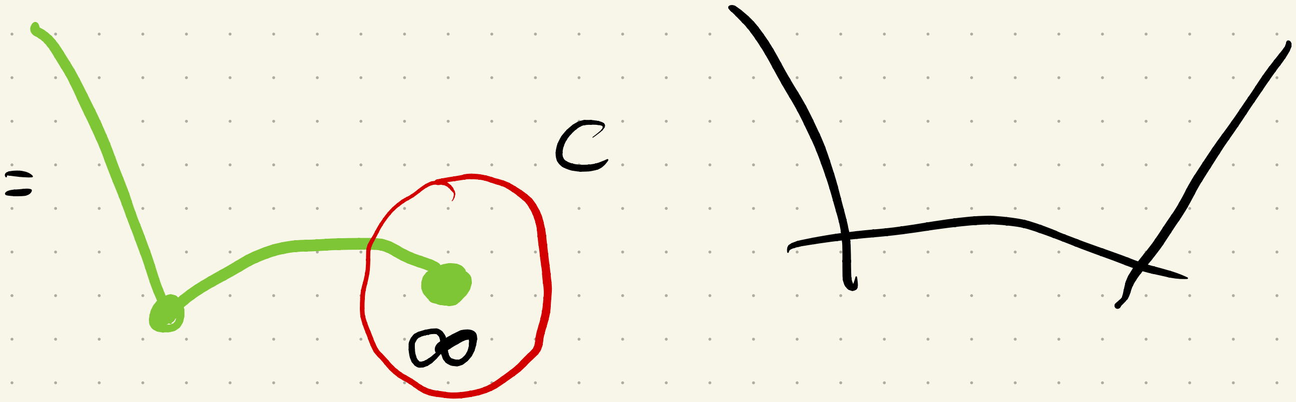


$$\mathrm{Attr}_\sigma(\infty) = \mathbb{P}^1 \setminus 0.$$

$$\mathrm{Attr}_\sigma(0) = N_0^\bullet$$

$$\mathrm{Attr}^f_r(p) = \bigcup_{q \leq p} \mathrm{Attr}_\sigma(q)$$

$\mathrm{Attr}^f_r(\infty) =$



Def: $\mathrm{Stab}_{\sigma, T^{1/2}, \mathcal{L}}(p) \in K_T(X)$

is the unique class such that

1) Supported on $\mathrm{Attr}^f_\sigma(p)$

2) $\mathrm{Stab}(p)|_p = \hat{\mathcal{L}} \otimes \mathcal{O}_{\mathrm{Attr}(p)}$

explicit line bundle

3) $\deg_A(\mathrm{stab}(p)|_q) \otimes \mathcal{L}_p < \deg_A(\mathrm{stab}(q)|_q) \otimes \mathcal{L}_q$

where $p, q \in X^A$ $q < p$

There are a number of proofs of existence & uniqueness of stable envelopes.

$$\mathrm{Stab}_{\sigma, T^{1/2}, \mathcal{L}} : K_T(X^A) \longrightarrow K_T(X)$$

satisfies some nice properties

1) $\mathrm{stab}(p)$, $p \in X^A$ is a basis of $K_T(X)_{loc}$. (~ Equivariant localization)

2) $\langle \mathrm{stab}_{aux}(p), \mathrm{stab}_{-aux}(q) \rangle = \delta_{p,q}$.

We will see how stab allows us to study quasimap invariants [AO].

We will need an elliptic version of stab.

Equiv. <u>Elliptic cohomology</u> of X

Fix an elliptic curve $E = \mathbb{C}^\times / q^{\mathbb{Z}}$ $\quad 0 < |q| < 1$

Given a torus $A \simeq (\mathbb{C}^\times)^n$, let

$\mathcal{E}_A = A / q^{X_0(A)}$ rk n abelian variety

Elliptic cohomology is a covariant functor

$$\mathrm{Ell}_A(-) : A\text{-}\mathrm{Sch} \longrightarrow \mathrm{Sch}.$$

$$\mathrm{Ell}_A(pt) = \mathcal{E}_A.$$

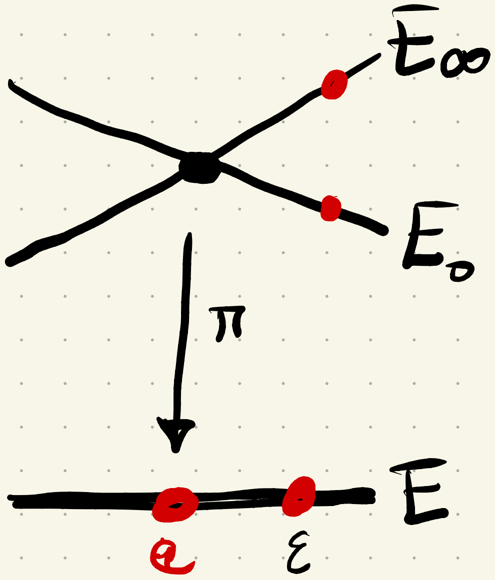
Analogy with K-theory:

$$\mathrm{Spec}\big(K_A(-)\big) : A\text{-}\mathrm{Sch} \longrightarrow \mathrm{Sch}.$$

$$\mathrm{Spec}(K_A(pt)) = A.$$

__Example:__ $X = \mathbb{P}^1 \curvearrowleft \mathbb{T}^1 = A$.

$$\begin{array}{c} Ell_A(X) \\ \Big\downarrow \pi \\ Ell_A(pt) \end{array} = E$$



$\infty \in X^A$

$0 \in X^A$

$\pi^{-1}(\varepsilon) = K_A(X)_\varepsilon \simeq K_A(X^A)$

$\pi^{-1}(e) = K(X)$

Elliptic stable envelopes [AO]:

Might expect: $H^0(\mathrm{Ell}_T(X^A), P_1) \xrightarrow{\text{stab}} H^0(\mathrm{Ell}_T(X), P_2)$

line bundles

Recall that K-th. stable envelope depends on a slope ($\in \mathrm{Pic}(X) \otimes \mathbb{R}$).

Elliptic stable envelope depends continuously on this parameter:

Actually: $H^0(\mathrm{Ell}_T(X^A) \times E_{\mathrm{Pic}(X)}, P) \longrightarrow$ same for X.

extra factor: abelian variety of Kähler parameters

In practice, for simple examples, such as $T^*\mathbb{P}^1$, can write explicit formulas for stab in terms of theta functions.

Application to quasimaps:

"Elliptic R-matrix"

$$R = \mathrm{Stab}_{\sigma'}^{-1} \circ \mathrm{Stab}_{\sigma} \circlearrowright H^0(\mathrm{Ell}_T(X^A) \times E_{\dots}, \mathcal{L})_{loc}$$

Can express the monodromy of quantum difference equations satisfied by vertex functions in terms of R.

For more details, see Aganagic & Okounkov's Elliptic Stable Envelopes.