Tensionless AdS/CFT

Matthias Gaberdiel
ETH Zürich

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Based mainly on work with
Rajesh Gopakumar and Lorenz Eberhardt

Plan of talk

1. Introduction and Motivation
2. Higher Spin symmetry
3. **The spectrum for AdS3/CFT2 I: NS-R**
4. The spectrum for AdS3/CFT2 II: Hybrid
5. Matching correlators in AdS3/CFT2
6. Generalisation to AdS5

NS-R WZW model

The perturbatively solvable world-sheet theory for AdS3 is formulated in terms of a WZW model based on the Lie algebra sl(2,R). For the case with supersymmetry the relevant algebra is

[Maldacena, (Son), Ooguri ’00 & ‘01]

$$\mathfrak{sl}(2,\mathbb{R})^{(1)}_k \cong \mathfrak{sl}(2,\mathbb{R})_{k+2} \oplus 3 \text{ free fermions}$$

bosonic: $J^3_n, J^\pm_n$

decoupled

The free fermions sit in the usual NS/R representations.

Representations

The highest weight representations of the $\mathfrak{sl}(2,\mathbb{R})_k$ affine algebra are of the form

$$J^{a_1}_{-n_1} \cdots J^{a_l}_{-n_l} |j, m\rangle$$

characterised by rep of the $\mathfrak{sl}(2,\mathbb{R})$ zero mode algebra

Geometric considerations (large level) suggest that the relevant representations should be of two kinds:

NS-R WZW model

Discrete series lowest weight reps:

$$\mathcal{D}_j^+ : \qquad C = -j(j-1) \ , \quad J_0^- |j,j\rangle = 0$$

Continuous series reps:

$$C_\alpha^j : \quad C = -j(j-1) = \tfrac{1}{4} + p^2 \ , \quad |j,m\rangle \text{ with } m \in \alpha + \mathbb{Z}$$

$$(j = \tfrac{1}{2} + ip)$$

[Maldacena, Ooguri ’00]

No-ghost theorem

Because of the Maldacena-Ooguri (unitarity) bound,

$$\text{MO-bound}: \qquad \frac{1}{2} < j < \frac{k+1}{2}$$

[Petropoulos '90]
[Hwang '91]
[Evans, MRG, Perry '98]
[Maldacena, Ooguri '00]

the (discrete) spectrum is bounded from above. Additional states are spectrally flowed images of these two classes of representations

They are not Virasoro highest weight, and are therefore best described in terms of the spectral flow w.

[Maldacena, Ooguri '00]
see also [Henningson et.al. '91]

Spectral flow automorphism

Basic idea: work with original highest weight rep. space, but define on it a new action (by automorphism):

$$J^3_m = \tilde{J}^3_m + \frac{kw}{2}\,\delta_{m,0}$$

$$J^\pm_m = \tilde{J}^\pm_{m\mp w}$$

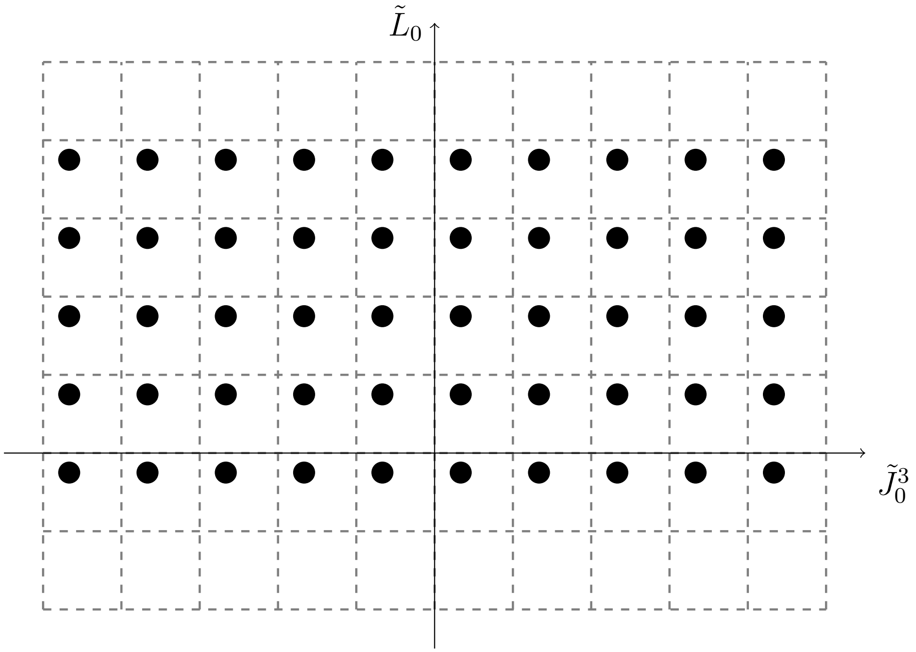
$$L_m = \tilde{L}_m - w\tilde{J}^3_m - \frac{k}{4}w^2\delta_{m,0}$$

Here the tilde modes act as in the original highest weight representation, but we think of action in terms of new un-tilde modes.

Spectral flow

In order to get a sense of what this means, let's concentrate on a continuous series rep for the case k=w=1:

$$J^3_m = \tilde{J}^3_m + \tfrac{1}{2}\,\delta_{m,0}$$
$$J^\pm_m = \tilde{J}^\pm_{m\mp1}$$
$$L_m = \tilde{L}_m - \tilde{J}^3_m - \tfrac{1}{4}\delta_{m,0}$$



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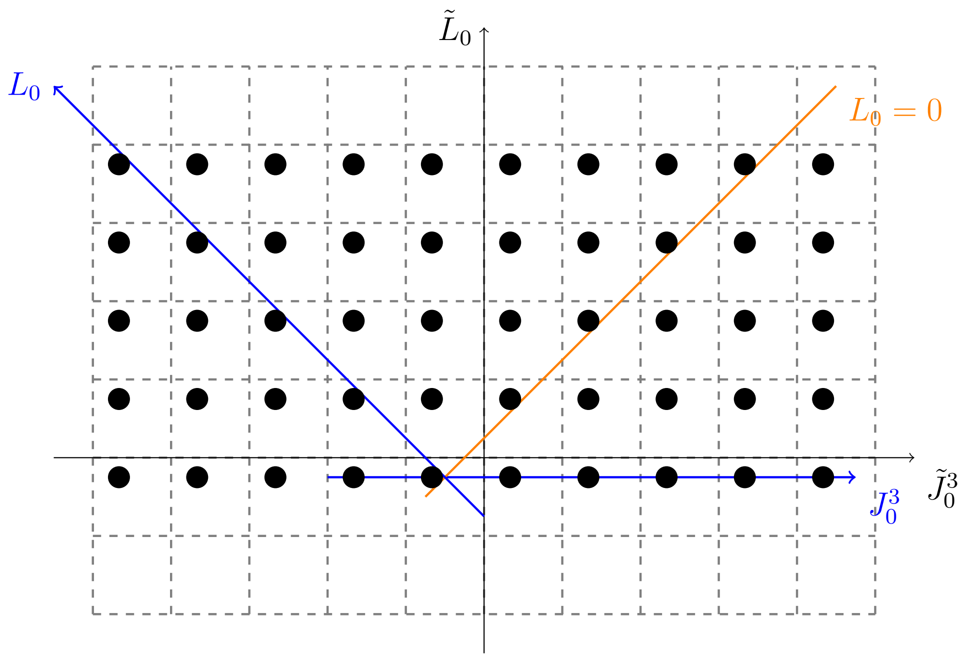

Spectral flow

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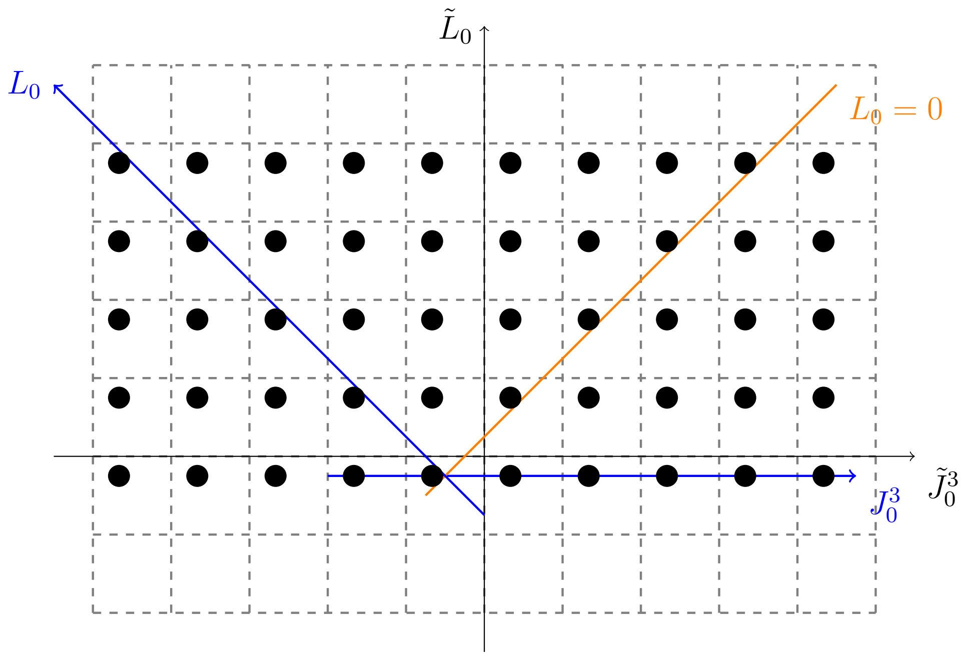


Thus the spectrally flowed representation is not Virasoro highest weight, i.e. the L_0 spectrum is unbounded from below — analogous to string theory on flat Minkowski space.

Physical states

This description is covariant, i.e. we need to impose the physical state condition, e.g. in NS sector

$$G_r^{\mathrm{tot}}\Phi = 0 \quad (r>0)$$

$$(L_0^{\mathrm{tot}} - \tfrac{1}{2})\Phi = 0 \ .$$

The second condition (mass-shell condition) implies that e.g. in sector without spectral flow

$$\frac{C}{k} + h_0 + N = \frac{1}{2} \qquad (\text{NS-sector})$$

Casimir of sl(2,R)

World-sheet conformal dim. of internal CFT

Dual CFT

The dual (`spacetime') CFT lives on the boundary of AdS3, and we have the identifications

$$L_0^{\mathrm{CFT}} = J_0^3 \ , \qquad L_1^{\mathrm{CFT}} = J_0^- \ , \qquad L_{-1}^{\mathrm{CFT}} = J_0^+ \ ,$$

with a similar relation for the right-movers.

We are interested in the `tensionless' regime of this theory. Since the level k is proportional to the size of the AdS3 space in string units, this should correspond to smallest (non-trivial) value of k: k=1.

Dual CFT

Thus we are led to study the physical spectrum of the (spacetime) theory for k=1 systematically.

As we shall see, the interesting part of the spectrum comes from the spectrally flowed continuous series reps.

Continuous reps

For the spectrally flowed continuous series reps, the mass-shell condition (in the NS sector) is at k=1

$[L_n = \tilde{L}_n - w\tilde{J}^3_n - \frac{k}{4}w^2\delta_{n,0}]$

$[\tilde{L}_0 = \frac{C}{k} + h_0 + N]$

$$C - wm - \frac{1}{4}w^2 + N = \frac{1}{2} \qquad \text{where } C = \tfrac{1}{4} + p^2$$

Here m is the $\tilde{J}^3_0$ eigenvalue (i.e. before spectral flow), and we have set $h_0 = 0$ (for simplicity).

For the continuous series rep we can simply solve this equation for m. For the case of p=0 we then get

Continuous reps

$$C - wm - \frac{1}{4}w^2 + N = \frac{1}{2} \qquad \text{with } C = \tfrac{1}{4}$$

$$m = \frac{1}{w}\left[N - \frac{w^2 + 1}{4}\right]$$

Continuous reps

$$C - wm - \frac{1}{4}w^2 + N = \frac{1}{2} \qquad \text{with } C = \tfrac{1}{4}$$

$$m = \frac{1}{w}\left[N - \frac{w^2+1}{4}\right]$$

Then observing that the actual J_0^3 eigenvalue is

$[J_n^3 = \tilde{J}_n^3 + \frac{wk}{2}\delta_{n,0}]$

$$h = m + \frac{w}{2} = \frac{N}{w} + \frac{w^2-1}{4w}\ .$$

Full spectrum

$$h = m + \frac{w}{2} = \frac{N}{w} + \frac{w^2-1}{4w} \, .$$

w-twisted modes

ground state energy in
w-twisted sector

Symmetric orbifold formula for cycle length w!

[MRG, Gopakumar ’18]
[Giribet, Hull, Kleban, Porrati, Rabinovici ’18]
see also [Giveon, Kutasov, Rabinovici, Sever ’05].

Note that for w=1 and N=0, this includes in particular chiral states (h=0) that correspond to massless higher spin fields!

[MRG, Gopakumar, Hull ’17]
[Ferreira, MRG, Jottar ’17]

Symmetric orbifold basics

Recall basic structure of symmetric orbifold

$$(\mathbb{T}^4)^N / S_N$$

untwisted sector: permutation invariant combinations

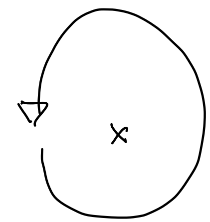
twisted sectors: associated to conjugacy classes of S_N

labelled by cycle shapes, i.e. partitions of N

concentrate on **single cycle sectors** $\longleftrightarrow$ analogue of single trace

Single cycle twisted sector

Consider bosons in w-cycle twisted sector:



$\partial X^a \to \partial X^{a+1}$

$a=1,\dots,w-1$

$\partial X^w \to \partial X^1$

Define: $\partial X^{(\ell)} = \sum_{a=1}^{w} \partial X^a \, e^{2\pi i a\ell/w}$

$$\partial X^{(\ell)} = \sum_r \alpha_r^{(\ell)} z^{-r-1}$$

$$\partial X^{(\ell)} \to e^{-2\pi i \ell/w} \partial X^{(\ell)}$$

$$\partial X^{(\ell)}(e^{2\pi i} z) = \sum_r \alpha_r^{(\ell)} z^{-r-1} \underbrace{e^{-2\pi i r}}_{= e^{-2\pi i \ell/w}}$$

$\therefore \; r = \frac{\ell}{w} \mod 1$

Casimir energy: $\frac{c}{24} \frac{w^2-1}{w}$

Full symmetric orbifold

For $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$ at k=1, criticality implies that the bosonic su(2) factor appears at level -1, and thus the analysis in the NS-R sector is a bit formal — in the hybrid formalism this will be cleaner (see below).

In order to get a sense of what will happen, we can use that

$$\mathfrak{su}(2)_{-1} \oplus \mathfrak{u}(1) = 4 \text{ symplectic bosons}$$

[Goddard, Olive, Waterson '87]

The 4 symplectic bosons behave as ghosts (on the level of the partition function) and remove 4 of the 8 fermions.

Which orbifold

Counting: $sl(2,\mathbb{R})$ 3 bos + 3 fer − (2+2) : 1 bos + 1 fer

$su(2)$ ≃ 4 symplectic bosons + 3 fermions

T^4 = 4 bos + 4 fer

This therefore suggests that we end up with 4+4 free bosons and fermions, i.e. with the spectrum of

$$\boxed{\text{symmetric orbifold of } \mathbb{T}^4}$$

[MRG, Gopakumar '18]

Continuum of states

However, the spectrum still seems to have a continuum (we earlier set p=0 by hand), which is not present in the symmetric orbifold theory.

There are also some discrete series rep states that do not fit into the above.

Thus we have not quite managed yet to identify the world-sheet theory that corresponds to the symmetric orbifold.

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Hybrid formalism

[Berkovits, Vafa, Witten '99]

In the hybrid formalism the world-sheet theory is described (for pure NS-NS flux) by the WZW model based on

$$\mathfrak{psu}(1,1|2)_k$$

together with the (topologically twisted) sigma model for T4. For generic k, this description agrees with the NS-R description a la Maldacena Ooguri.

[Troost '11], [MRG, Gerigk '11]
[Gerigk '12]

Supergroup

The supergroup $\mathrm{PSU}(1,1|2)$ has the basic form

$$\begin{pmatrix} \mathrm{SL}(2,\mathbb{R}) & \text{ferm gen.} \\ \text{ferm gen.} & \mathrm{SU}(2) \end{pmatrix} \qquad [\mathrm{SL}(2,\mathbb{R}) \cong \mathrm{SU}(1,1)]$$

Corresponding Lie algebra generators

$$\begin{aligned} \mathfrak{sl}(2,\mathbb{R}) : &\qquad J^a \\ \mathfrak{su}(2) : &\qquad K^a \\ \text{ferm gen.} : &\qquad S^{\alpha\beta\gamma} \end{aligned}$$

Supergroup

The super Lie algebra $\mathfrak{psu}(1,1|2)_k$ is

$$
\begin{aligned}
[J^3_m, J^3_n] &= -\tfrac{1}{2} k m \delta_{m+n,0} \ , \\
[J^3_m, J^\pm_n] &= \pm J^\pm_{m+n} \ , \\
[J^+_m, J^-_n] &= k m \delta_{m+n,0} - 2 J^3_{m+n} \ , \\
[K^3_m, K^3_n] &= \tfrac{1}{2} k m \delta_{m+n,0} \ , \\
[K^3_m, K^\pm_n] &= \pm K^\pm_{m+n} \ , \\
[K^+_m, K^-_n] &= k m \delta_{m+n,0} + 2 K^3_{m+n} \ , \\
[J^a_m, S^{\alpha\beta\gamma}_n] &= \tfrac{1}{2} c_a (\sigma^a)^\alpha{}_\mu S^{\mu\beta\gamma}_{m+n} \ , \\
[K^a_m, S^{\alpha\beta\gamma}_n] &= \tfrac{1}{2} (\sigma^a)^\beta{}_\nu S^{\alpha\nu\gamma}_{m+n} \ , \\
\{S^{\alpha\beta\gamma}_m, S^{\mu\nu\rho}_n\} &= k m \epsilon^{\alpha\mu} \epsilon^{\beta\nu} \epsilon^{\gamma\rho} \delta_{m+n,0} - \epsilon^{\beta\nu} \epsilon^{\gamma\rho} c_a \sigma^{\alpha\mu}_a J^a_{m+n} + \epsilon^{\alpha\mu} \epsilon^{\gamma\rho} \sigma^{\beta\nu}_a K^a_{m+n} \ .
\end{aligned}
$$

$J^a_0 \leftrightarrow L_0, L_{\pm 1}$

$K^a_0 \leftrightarrow$ su(2) zero modes

$S^{\alpha\beta\gamma}_0 \leftrightarrow G^\pm_{\pm 1/2}, \ G'^\pm_{\pm 1/2}$

global $d=2$ $\mathcal{N}=4$ superconf. algebra

bosonic subalgebra

$sl(2,\mathbb{R})_k \oplus su(2)_k$.

Hybrid formalism

For the following it will be important to understand the representation theory of

$$\mathfrak{psu}(1,1|2)_1$$

The bosonic subalgebra of this superaffine algebra is

$$\mathfrak{sl}(2)_1 \oplus \mathfrak{su}(2)_1$$

Thus only **n**=1 and **n**=2 are allowed for the highest weight states.

Short representations

A generic representation of the zero mode algebra $\mathfrak{psu}(1,1|2)$ has the form

rep of sl(2,R)

$$(C^j_\alpha, \mathbf{n})$$

$$(C^{j+\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n+1}) \qquad (C^{j+\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n-1}) \qquad (C^{j-\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n+1}) \qquad (C^{j-\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n-1})$$

$$(C^{j+1}_\alpha, \mathbf{n}) \qquad (C^j_\alpha, \mathbf{n+2}) \qquad 2\cdot(C^j_\alpha, \mathbf{n}) \qquad (C^j_\alpha, \mathbf{n-2}) \qquad (C^{j+1}_\alpha, \mathbf{n})$$

$$(C^{j+\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n+1}) \qquad (C^{j+\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n-1}) \qquad (C^{j-\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n+1}) \qquad (C^{j-\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n-1})$$

$$(C^j_\alpha, \mathbf{n})$$

rep of su(2) of dim = n+1.

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$$(C^{j+\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n+1}) \quad (C^{j+\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n-1}) \quad (C^{j-\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n+1}) \quad (C^{j-\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{n-1})$$

$$(C^j_\alpha, \mathbf{n})$$

Thus for k=1 need a short rep!

Short representations

In fact, the only representations that are allowed are

$$(C^{j}_{\alpha}, \mathbf{2})$$

$$(C^{j+\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{1}) \qquad (C^{j-\frac{1}{2}}_{\alpha+\frac{1}{2}}, \mathbf{1})$$

and the shortening condition actually implies that this is only possible provided that

$$j = \frac{1}{2} \quad \longrightarrow \quad \textbf{NO CONTINUUM!}$$

[Eberhardt, MRG, Gopakumar '18]

Free field realisation

For the following it will be useful to describe $\mathfrak{psu}(1,1|2)_1$ in terms of free fields.

[Eberhardt, MRG, Gopakumar ’18]

The relevant free fields are

4 symplectic bosons, i.e. 2 $\text{spin-}\frac{1}{2}\ \beta\gamma$ systems

$$[\xi^\alpha_r, \eta^\beta_s] = \epsilon^{\alpha\beta}\, \delta_{r,-s} \qquad\qquad (\alpha, \beta \in \{\pm\})$$

4 free fermions

$$\{\psi^\alpha_r, \chi^\beta_s\} = \epsilon^{\alpha\beta}\, \delta_{r,-s} \qquad\qquad (\alpha, \beta \in \{\pm\})$$

Free field realisation

In terms of these free fields the current generators are:

$$\boxed{\begin{aligned} [\xi^\alpha_r, \eta^\beta_s] &= \epsilon^{\alpha\beta}\, \delta_{r,-s} \\ \{\psi^\alpha_r \chi^\beta_s\} &= \epsilon^{\alpha\beta}\, \delta_{r,-s} \end{aligned}}$$

$$\begin{aligned}
J^3_m &= -\tfrac{1}{2}(\eta^+\xi^-)_m - \tfrac{1}{2}(\eta^-\xi^+)_m \,, & K^3_m &= -\tfrac{1}{2}(\chi^+\psi^-)_m - \tfrac{1}{2}(\chi^-\psi^+)_m \,, \\
J^\pm_m &= (\eta^\pm\xi^\pm)_m \,, & K^\pm_m &= \pm(\chi^\pm\psi^\pm)_m \,, \\
S^{\alpha\beta+}_m &= (\chi^\beta\xi^\alpha)_m \,, & S^{\alpha\beta-}_m &= -(\eta^\alpha\psi^\beta)_m \,, \\
U_m &= -\tfrac{1}{2}(\eta^+\xi^-)_m + \tfrac{1}{2}(\eta^-\xi^+)_m \,, & V_m &= -\tfrac{1}{2}(\chi^+\psi^-)_m + \tfrac{1}{2}(\chi^-\psi^+)_m \,.
\end{aligned}$$

In order to obtain actually $\mathfrak{psu}(1,1|2)_1$ need to gauge by

$$Z_m = U_m + V_m \cong 0$$

R sector representation

The ground states of the R-sector representation transform under the zero modes as

$$
\begin{aligned}
\xi_0^+ |m_1, m_2\rangle &= |m_1, m_2 + \tfrac{1}{2}\rangle \,, & \eta_0^+ |m_1, m_2\rangle &= 2\, m_1\, |m_1 + \tfrac{1}{2}, m_2\rangle \,, \\
\xi_0^- |m_1, m_2\rangle &= -\, |m_1 - \tfrac{1}{2}, m_2\rangle \,, & \eta_0^- |m_1, m_2\rangle &= -2\, m_2\, |m_1, m_2 - \tfrac{1}{2}\rangle \,, \\
\chi_0^+ |m_1, m_2\rangle &= 0 \,, & \psi_0^+ |m_1, m_2\rangle &= 0 \,.
\end{aligned}
$$

The relevant charges are then

$$
\begin{aligned}
J_0^3\, |m_1, m_2\rangle &= (m_1 + m_2)\, |m_1, m_2\rangle \\
K_0^3\, |m_1, m_2\rangle &= \tfrac{1}{2}\, |m_1, m_2\rangle & L_0\, |m_1, m_2\rangle &= 0 \\
U_0\, |m_1, m_2\rangle &= (m_1 - m_2 - \tfrac{1}{2})\, |m_1, m_2\rangle & V_0\, |m_1, m_2\rangle &= 0
\end{aligned}
$$

R sector representation

The relevant charges are then

$$J_0^3 \, |m_1, m_2\rangle = (m_1 + m_2) \, |m_1, m_2\rangle$$

$$K_0^3 \, |m_1, m_2\rangle = \tfrac{1}{2} \, |m_1, m_2\rangle \qquad L_0 \, |m_1, m_2\rangle = 0$$

$$U_0 \, |m_1, m_2\rangle = (m_1 - m_2 - \tfrac{1}{2}) \, |m_1, m_2\rangle \qquad V_0 \, |m_1, m_2\rangle = 0$$

$C = \frac{1}{4} - U_0^2 = -j(j-1)$ $\qquad j = m_1 - m_2 \qquad m = m_1 + m_2$

On ground state $|m_1, m_2\rangle$: $\quad U_0 + V_0 = 0 = m_1 - m_2 - \frac{1}{2} \qquad j = m_1 - m_2 = \frac{1}{2}$

$\left(C_\alpha^{1/2}, \underline{2}\right) \qquad \alpha = m_1 + m_2 \pmod 1$.

Spectral flow

The full worldsheet spectrum consists of this R-sector representation, together with its spectrally flowed images. Here spectral flow comes from

[Henningson et.al. '91]
[Maldacena, Ooguri '00]

$$\tilde{\xi}_n^{\pm} = \xi^{\pm}_{n\pm\frac{w}{2}} \ , \qquad \tilde{\eta}_n^{\pm} = \eta^{\pm}_{n\pm\frac{w}{2}} \ , \qquad \tilde{\psi}_n^{\pm} = \psi^{\pm}_{n\mp\frac{w}{2}} \ , \qquad \tilde{\chi}_n^{\pm} = \chi^{\pm}_{n\mp\frac{w}{2}} \ ,$$

consider above R sector rep. for tilde modes $\longrightarrow$ interpret in terms of `untilde' modes

As before, for w>1 not highest weight representation any longer.

Spectral flow

Under spectral flow the different generators transform as

$$
\begin{aligned}
J^3_m &= \tilde{J}^3_m + \frac{kw}{2}\delta_{m,0} \ , \\
J^\pm_m &= \tilde{J}^\pm_{m\mp w} \ , \\
K^3_m &= \tilde{K}^3_m + \frac{kw}{2}\delta_{m,0} \ , \\
K^\pm_m &= \tilde{K}^\pm_{m\pm w} \ , \\
S^{\alpha\beta\gamma}_m &= \tilde{S}^{\alpha\beta\gamma}_{m+\frac{1}{2}w(\beta-\alpha)} \ , \\
L_m &= \tilde{L}_m + w(\tilde{K}^3_m - \tilde{J}^3_m) \ , \\
Z_m &= \tilde{Z}_m \ .
\end{aligned}
$$

$$\tilde{J}^+_m = \sum_r :\tilde{\xi}^+_r \tilde{\eta}^+_{m-r}:$$

$$= \sum_r :\xi^+_{r+\frac{w}{2}} \eta^+_{m-r+\frac{w}{2}}: = J^+_{m+w}$$

$$\therefore \ J^+_m = \tilde{J}^+_{m-w}$$

Mass-shell condition: $L_0\phi = 0$

Constraint equation: $Z_n\phi = 0 \quad (n \geq 0)$

Physical states (bosons)

Bosonic oscillators:

$$\begin{aligned} \mathbb{T}^4 &: \quad 4 \text{ bosons} \\ \mathfrak{u}(1,1|2)_1 &: \quad 4 \text{ symplectic bosons} \end{aligned}$$

Physical state conditions:

$$\begin{aligned} Z_n = 0 &\;\rightarrow\; \text{ removes } 2 \text{ bosons} \\ L_n = 0 &\;\rightarrow\; \text{ removes } 2 \text{ bosons} \end{aligned}$$

Thus only the 4 torus bosons, say, survive.

Physical states (bosons)

Consider thus the state $\alpha^{i_1}_{-n_1} \cdots \alpha^{i_l}_{-n_l} |m_1, m_2\rangle$

Zero mode conditions:

$$Z_0 = 0 \;\; \rightarrow \quad m_1 - m_2 = \tfrac{1}{2}$$

$$L_0 = 0 \;\; \rightarrow \quad N + w(\tilde{K}^3_0 - \tilde{J}^3_0) = N + w(\tfrac{1}{2} - \tilde{J}^3_0) = 0$$

Thus spacetime conformal dimension is

$$\boxed{J^3_0 = \tilde{J}^3_0 + \frac{w}{2} = \frac{N}{w} + \frac{w+1}{2}}$$

Physical states (bosons)

$$\boxed{J_0^3 = \tilde{J}_0^3 + \frac{w}{2} = \frac{N}{w} + \frac{w+1}{2}}$$

Thus we have the correspondence

$$\alpha^{i_1}_{-n_1} \cdots \alpha^{i_l}_{-n_l} |m_1, m_2\rangle \quad \longleftrightarrow \quad \alpha^{i_1}_{-\frac{n_1}{w}} \cdots \alpha^{i_l}_{-\frac{n_l}{w}} |\mathrm{BPS}\rangle_{h=\frac{w+1}{2}}$$

Analysis for fermions is similar, and we thus get exactly the (single-particle) spectrum of

$$\mathrm{Sym}_N(\mathbb{T}^4)$$

in the large N limit.

[Eberhardt, MRG, Gopakumar '18]