Maths Circle Explorations: Session 3

November 26, 2021

1. A new test has been developed for a rare disease. The test is 99% accurate. This means that 99 out of 100 times, the test correctly identifies whether or not a person has the disease. Studies have estimated that, at any given time, this disease affects about 1 in 10000 people in the general population. Uma decides to get tested for the disease.

   (a) Suppose that the test result is positive (i.e., the test says that Uma has the disease). What is the probability that Uma actually has the disease? If the test result is negative, then what is the probability that Uma really does not have the disease?

   (b) How do your answers change if the disease is less rare, and is found in 1% of the general population?

2. The object shown on the left in Figure 1 is made of a material that can be stretched, compressed, bent, distorted and molded at will. That is, one can change its shape in any way one pleases. However, one cannot tear the material, or stick two parts of it together. Convince yourself that it is possible to deform the object on the left in Figure 1 so that it is transformed into the object shown on the right in the same figure. Draw a "movie" (a sequence of stills) that shows this deformation process unfold.

   Make linked rings by joining your index fingers with your thumbs. Observe that what you have demonstrated above is the following: if the human body were elastic enough, it would be possible to move your hands apart without ever separating the joined fingertips.

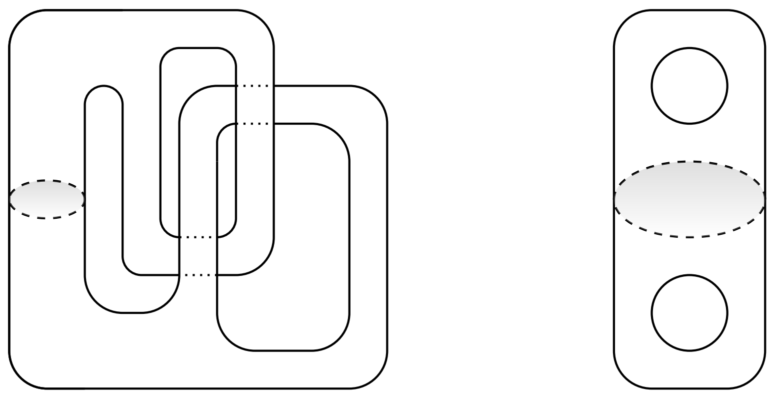

Figure 1: Can the rings be unlinked?

3. (a) Convince yourself that the bagel shaped object in Figure 2 below can be constructed by taking the sheet of elastic material shown on the right, and gluing together the edges with identical labels (in such a way that the directions of the arrows on the edges match up). Compare this with Activity 2 of Session 1, and the doughnut shaped chessboard of Activity 3 in Session 2.

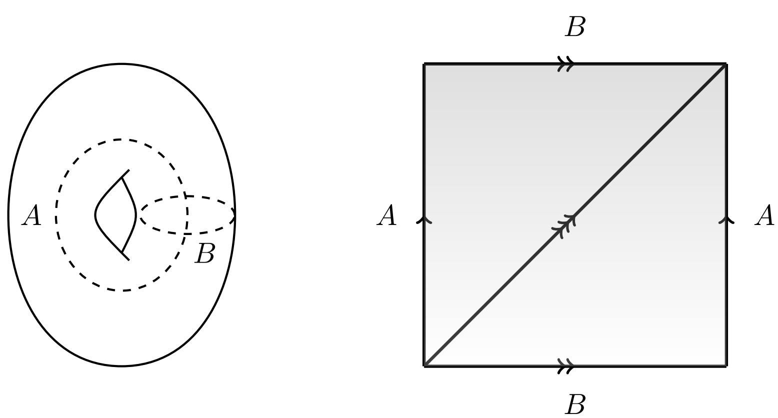

Figure 2: Constructing a torus from a sheet of elastic material

(b) Figure 3 shows "pretzel-like" surfaces with 2 "holes" and 3 holes. Can you think of a way to construct these surfaces that is analogous to the construction of the surface of a bagel in part (*a*)? That is, can you find a way to cut out a shape from a thin sheet of elastic material, and glue together various boundary edges to get the shapes shown in Figure 3? Does your construction work if the pretzel has 4 or more holes?

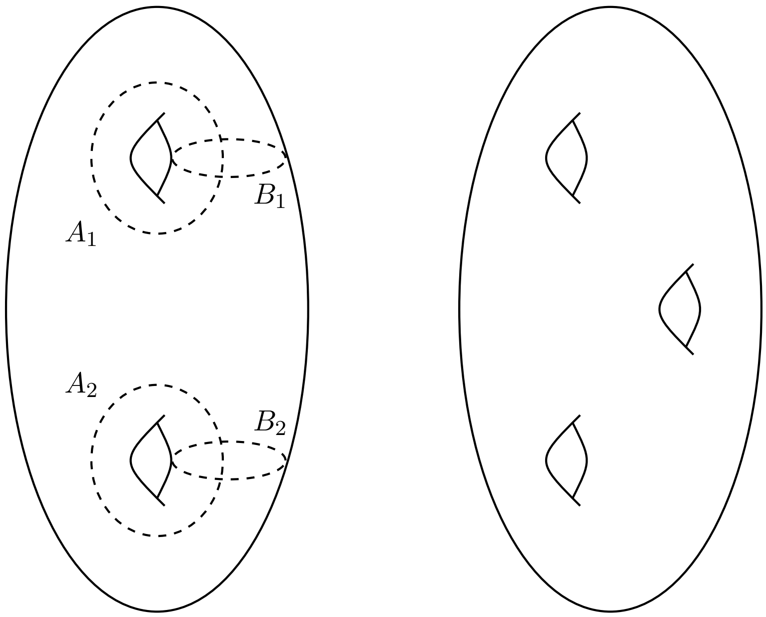

Figure 3: The surface of a bagel with extra "handles" attached.

(c) Find a way to cut up the surfaces in part (b) into triangles, as was done for the bagel in part (a). Count the number of vertices, edges and triangles that you get. How are they related to the number of holes in the pretzel? How does your answer change if you chop up the surface into triangles in a different way?

(d) Suppose that the material that the pretzels are made of behaves exactly like the material described in Activity 2 above. Is it possible to deform the pretzel with 2 holes into a bagel? Can one deform it into a pretzel with 3 holes? Give a convincing argument to justify your answer.

4. Josh and Ishan are playing a game. The goal of the game is to find a transformation of the Euclidean plane onto itself that leaves distances unchanged, and carries a given triangle ABC onto another given triangle $A'B'C'$. Examples of transformations that leave distances unchanged include:

   - Rotation counterclockwise about a point O through an angle θ.

- Reflection in a line l. This is a transformation that does not move the points on l and moves every other point. For example, if l is the x-axis, then reflection in l sends the point (x, y) to $(x, -y)$.
- Translation by a vector $v = (\alpha, \beta)$. This sends the point (x, y) to the point $(x + \alpha, y + \beta)$.

Josh claims that given triangles ABC and $A'B'C'$ with the same side lengths, he can find a finite sequence of lines such that successively performing reflections in those lines takes the first triangle to the second. In order to challenge Josh, Ishan starts with a triangle ABC and constructs

- A triangle $A'B'C'$ by rotating the triangle ABC counterclockwise through an angle θ about a point O.
- A triangle DEF by translating ABC by a non-zero vector v.

(a) Can Josh succeed in transforming ABC into $A'B'C'$ and DEF using only a series of reflections in lines? Justify your answer.

(b) Josh constructs a new triangle UVW by rotating counterclockwise the triangle $A'B'C'$ constructed by Ishan through an angle ϕ about a point O'. Ishan claims that he can transform the triangle ABC into the triangle UVW using only a single rotation about a point O''. Is Ishan's claim true? If yes, what is the point O'', and what is the angle through which he must rotate ABC?