Maths Circle Explorations: Session 2

November 12, 2021

1. Belliappa and Gayatri left at dawn, one traveling from Hesaraghatta to Magadi, and the other from Magadi to Hesaraghatta. They were heading towards one another, each walking at a steady constant pace (along the same road). They met at noon, but did not stop, and each of them kept walking at the same speed as before. Gayatri arrived at Hesaraghatta at 4:00 pm, and Belliappa arrived at Magadi at 9:00 pm. At what time was dawn on that day?

2. Recall the two-player game from Activity 5 of Session 1. Perumal and Shehnaaz have been playing this game, in search of patterns. Based on their observations, Shehnaaz speculates that the initial configurations (n_1, n_2, n_3) of boxes can be organized into two types, L and W, such that

   (a) Each configuration is of *exactly one* of the two types.

   (b) $(0, 0, 0)$ is of type L.

   (c) If a configuration is of type L then *any* legal move will convert it into a configuration of type W.

   (d) If a configuration is of type W, then there is *at least one* legal move that converts it into a configuration of type L.

   She claims that if the initial configuration is of type L, then the second player has a winning strategy.

   Perumal has recently learned that for each natural number n, there is exactly one way to write it in the form

   $$n = a_0 + 2a_1 + 2^2a_2 + \ldots + 2^ka_k \quad (1)$$

   with $a_0, a_1, \ldots, a_k$ taking the values 0 or 1 (why is this true?). For example

   $$11 = 1 + 2(1) + 2^2(0) + 2^3(1).$$

   Through careful experiment Perumal and Shehnaaz have discovered that certain configurations (n_1, n_2, n_3) are losing positions for the first player. In a moment of inspiration, Perumal writes down the numbers (n_1, n_2, n_3) in these configurations in the form given in Equation 1. Staring at the table he has created, he is surprised to find a clear pattern emerge! He describes a rule that organizes the initial configurations into two types L and W.

(i) What is the rule that Perumal described?

(ii) Perumal and Shehnaaz are able to check that this rule does satisfy Shehnaaz's criteria $(a) - (d)$ above for small values of (n_1, n_2, n_3). However, they are unable to convince themselves that this rule works for very large numbers of boxes (large values of the numbers n_i). Do you believe the rule is correct? Can you convince them of your belief?

3. Did you enjoy Activity 4 of Session 1? Here are a few more puzzles involving knights and chessboards:

   (a) On a 4×4 chessboard, can you find a sequence of moves by which a knight visits each square on the board *exactly once*? The knight can start anywhere on the board, and the trip can end anywhere.

   (b) On a 3×3 chessboard, place 4 knights at the four corners; two white knights and two black knights. Can you find a sequence of moves by which the white knights swap places with the black knights?

   Experiment with variations on these themes. For instance, what changes if your chessboard is drawn on (and covers) the surface of a cylinder or a doughnut?

4. A very thin circular metal ring is heated unevenly by a collection of candles that are placed at various points along the ring. A few milliseconds after the heat source is removed, some parts of the ring are hot while others are cold. The only thing that we know is that at any given moment the temperature varies continuously: an ant walking along the ring would not experience an abrupt change in the temperature at any point. Armed with just this knowledge, Hana claims that there exist two points on the ring that are diametrically opposite to each other, and are also at exactly the same temperature. Is she correct? Try to find a watertight argument to convince others of your conclusions.