Random walk of an ant on the integer line: Imagine an ant walking randomly on the integer line. At every time step the ant takes one step, either to the right or to the left, with equal probability. We denote by X_n the position of the ant after n time steps. The ant's starting position is $X_0 = 0$ at time $n = 0$.

Problem A: Consider an $N = 4$ step walk that the walk takes.

1. How many distinct paths are there?
2. Write down all the distinct possible paths, using the notation that a right step is labelled as + and a left step is labelled as -. A possible 4-step walk is +-+-.
3. List all the possible final positions of the walk and use the answer of part 2 to find the probabilities of the ant being in the different positions.

Problem B: Consider an N step walk.

1. Any walk can be written as a string of "+" and "-" of length N. A given path will consist of R number of "+" steps and L number of "-" steps, with $N = R + L$. What is the number of distinct strings with a fixed number of Rs' and Ls' ? Use this to find the number of walks that end at a fixed final position. See if you can get the probabilities that you computed in Problem (A3)
2. Suppose we define random numbers V_n, $n = 1, 2,... N$, which are independent of each other and each have the following distribution: $V_n = 1$ with probability $1/2$ and $V_n = -1$ with probability $1/2$. Can we then write $X_n = X_{n-1} + V_n$, $n = 1, 2,... N$?
3. Use the above to show that, on average, $< X_N >= 0$ and $< X_N^2 >= N$.
4. Try to use the same approach as in (1) for an ant walking on a two-dimensional lattice. We want to find the probability of the ant being at some point (X,Y) after N steps.

Problem C: In the last class we computed the probability of the ant being at position X after N steps. This is given by $P(X, N) = \frac{N!}{(\frac{N+X}{2})! \, (\frac{N-X}{2})!} \frac{1}{2^N}$.

1. Use Desmos to plot this P(X,N) versus X, for N=32,64,128.
2. Plot $\sqrt{N} P(X, N)$ versus $X/\sqrt{N}$.
3. In (2), also add a plot of $\sqrt{2/\pi}\, e^{-X^2/2}$

Problem D: Watch a video of the Galton Board (https://www.youtube.com/watch?v=EvHiee7gs9Y) and see if you can explain the observed Bell curve and relate it to the previous problem.

Problem E: The ant has the worry "If I start walking randomly what is the probability that I will get back home!". Being mathematically inclined, it sets out to compute this probability. Help the ant in computing this.

1. What is the probability that the ant returns home, for the first time, after exactly 2 steps, 4 steps, 6 steps?
2. Compare this with the probability that the ant is at home after 2,4,6 steps.

Problem F: An ant walks on a two dimensional grid (or a square lattice). The position of the ant after N steps is given by the vector $Z_N = (X_N, Y_N)$, where $X_n = \sum_{n=1}^{N} x_n$ and $Y_n = \sum_{n=1}^{N} y_n$, and x_n, y_n are random variables which take values $+\ 1,\ -\ 1$ with equal probability. Write a code to show a few random trajectories of the ant for N=16 steps, given that the ant always starts from the origin. You can also do this on a graph paper by hand. Consider a thousand ants all staring from the origin and each doing their own random walk. Indicate the final position of all ants after N steps with a tiny circle. Plot the points for N=16, 64, 128 using different colors for the different time values. You will find that the ants are spread over a disc of radius R. Can you guess how R grows with N?