

# Exploration Sheet: Generating Functions in Combinatorial Problems

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## Introduction

Generating functions act as a bridge between discrete mathematics and continuous analysis. They are often described as a “clothesline on which we hang up a sequence of numbers for display.” In this sheet, you will explore how algebraic operations on polynomials and power series can elegantly model complex combinatorial distributions and partitions.

## 1. The Non-Empty Boxes (Identical Objects)

Let  $a_r$  be the number of ways of distributing  $r$  identical objects into  $n$  distinct boxes such that no box is empty.

**Explore:**

- **Constructing a Single Box:** If a single box must contain *at least one* object, write down an infinite series where the exponents of  $x$  represent the valid number of objects you can place in that box.
- **Combining Boxes:** What algebraic operation should you apply to your series to account for all  $n$  distinct boxes?
- **Algebraic Manipulation:** Write down the full generating function for  $a_r$ . How can you rewrite the geometric series portion to make it easier to expand?
- **Extraction:** By expanding your simplified generating function, find the general formula for the coefficient of  $x^r$ . How does this relate to the classic “stars and bars” combinatorial argument?

## 2. Partitions with Parity Constraints

A part in an integer partition is said to be *odd* if its size is an odd number. Let  $b_r$  denote the number of partitions of an integer  $r$  into only odd parts (for example, the partitions of 6 into odd parts are  $5 + 1$ ,  $3 + 3$ ,  $3 + 1 + 1 + 1$ , and  $1 + 1 + 1 + 1 + 1 + 1$ , so  $b_6 = 4$ ).

### Explore:

- **Isolating the Parts:** Consider just the 1s used in a partition. What infinite series represents the sum of choices for how many 1s you can include?
- **Scaling Up:** Now do the same for the 3s. Notice that choosing two 3s contributes 6 to the total sum. How does this change the exponents in your series for the 3s? What about the 5s?
- **The Infinite Product:** Construct the overall generating function for  $b_r$  by multiplying the series for all possible odd parts.
- **Interpretation:** Look at a specific term in the expansion. If you multiply  $(x^1)^5$  from the first factor and  $(x^3)^2$  from the second factor, what resulting power of  $x$  do you get, and what specific partition does this represent?

## 3. Restricted Sequences and Distinct Objects

For each  $r \in \mathbb{N}^*$ , we want to find  $a_r$ , the number of ways of distributing  $r$  *distinct* objects into 4 *distinct* boxes such that boxes 1 and 2 must hold an even number of objects, and box 3 must hold an odd number of objects. Box 4 has no restrictions.

### Explore:

- **Transforming the Problem:** First, justify why this distribution problem is exactly equivalent to finding the number of  $r$ -digit quaternary (base-4) sequences that contain an even number of 1s, an even number of 2s, and an odd number of 3s.
- **Choosing the Right Tool:** Because the objects (or the positions in the sequence) are *distinct*, ordinary generating functions will fail to account for permutations. What alternative type of generating function should you use here?
- **Modeling the Constraints:**
  - Write the generating function factor for a box that requires an *even* number of objects. (Hint: Think about combinations of  $e^x$  and  $e^{-x}$ ).
  - Write the generating function factor for a box that requires an *odd* number of objects.
  - Write the factor for the unrestricted box.
- **Synthesis:** Multiply these factors together. How can you expand the resulting product to eventually extract the coefficient that gives  $a_r$ ?