Math Circle session Aug 8, 2025

Approximate first, calculate later!

Question 1

Demonstrate how the 'squeeze' or 'sandwich' theorem works by using an Archimedean construction. Inscribe and circumscribe an N-gon within a unit circle to show the small angle approximation for sin θ and tan θ for large N (by 'squeezing the circle between the N-gons!)

Question 2

A physics application: Time period of a pendulum

Finding the period of a pendulum, even at small amplitudes, requires calculus because of the pendulum's varying speed. A *conical* pendulum consists of a bob of mass m suspended from a pivot by a string of length l (subtending an angle θ with the vertical at the pivot) which undergoes circular motion in the horizontal plane.
Projecting the motion of a conical pendulum (onto a vertical screen produces one-dimensional pendulum motion; thus, the period of the two-dimensional motion is the same as the period of one-dimensional pendulum motion! If you are familiar with using Newton's second law, use that idea to find the period of a pendulum using the small angle approximations you have proved above.

If you haven't yet seen Newton's second law, but you know about dimensional analysis, you can use another approach to this problem. Begin with a dimensional analysis proof of the dependence of the period on the length and acceleration due to gravity. Then find the dimensionless constant of 2π from the analysis above.

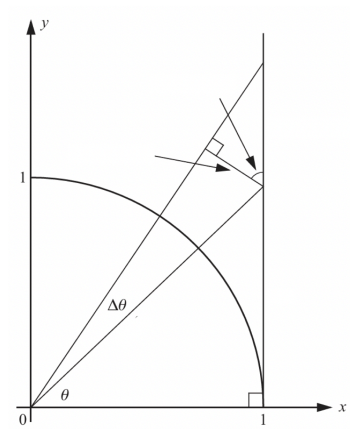
Question 3

A mathematics application: The derivative of the tangent

Using the picture below, construct a proof to show that

$\frac{d}{d\theta} tan\,\theta = sec^2\theta$

Hint: What are the angles and sides indicated with the arrows? Use similar triangles to determine an expression for $\frac{\Delta(\tan\theta)}{\Delta\theta}$ and then use the result for small angles you have derived in the not-so-distant past.



Just a little more physics…

A common analogy to Ohm's law is modeling the voltage as the height of a liquid above a hole in a cylindrical beaker, the current to the rate of flow and resistance inversely proportional to the area of the hole. However, there is a flaw in the analogy. Can you find it?

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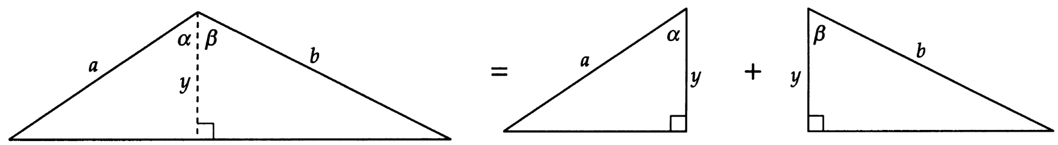
Pictorial proofs

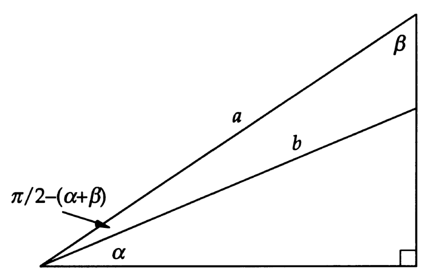
Warm up questions

1. Here is a picture to help you show that sine addition formula.
   $\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$

Fill in the missing angle and the lengths or areas to convince me that this picture is indeed a thousand words!



2. Use this picture to demonstrate the cosine sum $\cos(\alpha + \beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta$



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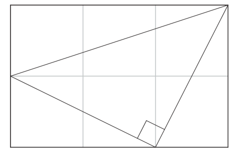
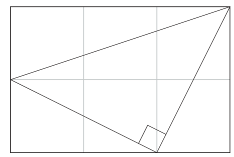
Question 1

One picture, several arc(ane!) tangent identities….
Indicate relevant angles on this figure to show that

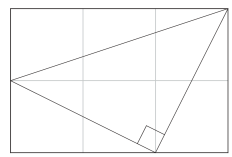
$$tan^{-1}\frac{1}{2} + tan^{-1}\frac{1}{3} = \frac{\pi}{4}$$



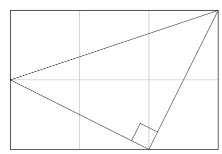
$$tan^{-1}3 \; - \; tan^{-1}\frac{1}{2} \; = \frac{\pi}{4}$$



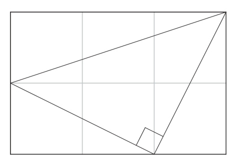
$$tan^{-1}2 \; - \; tan^{-1}\frac{1}{3} \; = \frac{\pi}{4}$$



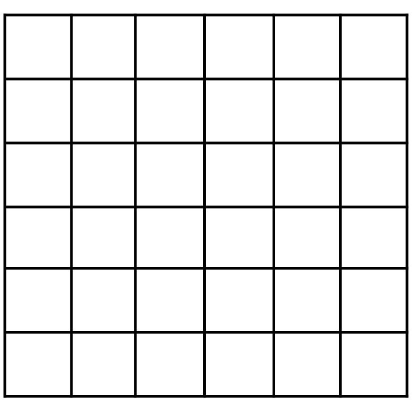
$$tan^{-1}1 \; + tan^{-1}\frac{1}{2} \; + tan^{-1}\frac{1}{3} \; = \frac{\pi}{2}$$



$$tan^{-1}1 \; + tan^{-1}2 \; + tan^{-1}3 \; = \pi$$



Draw another picture to show: $tan^{-1}1 \; + tan^{-1}2 \; + tan^{-1}3 \; = \pi$



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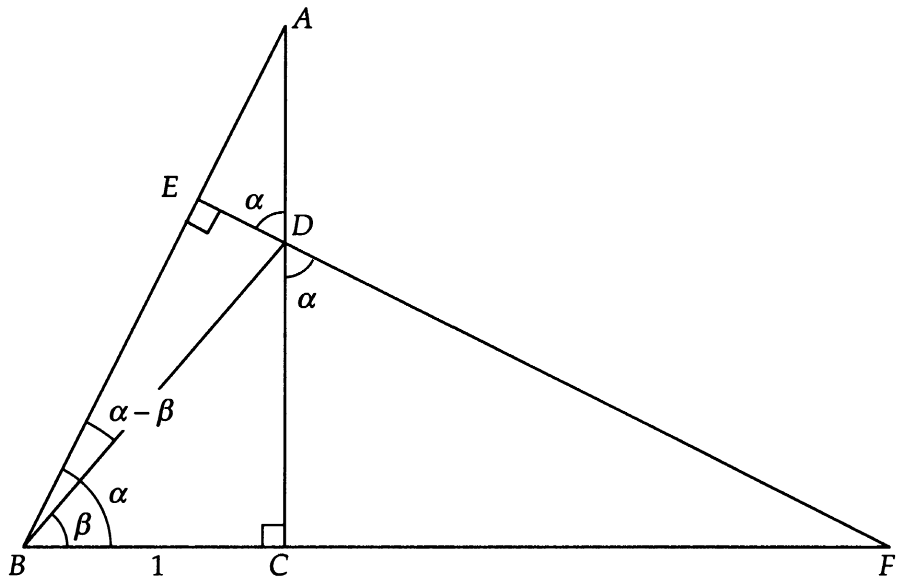
Question 2

Use the picture below and identify similar triangles to show the difference identity of tangents.
$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan\beta}{1+\tan \alpha \cdot \tan \beta}$



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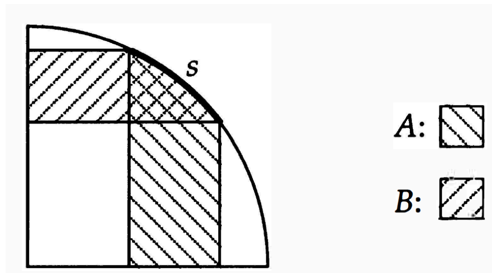
Question 3

Let s be any arc of a circle of radius r lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis, and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $A + B$ depends only on the arc length s and radius r, and not the position of s.



Question 4

Use the pictures below to show that the $V_{pyramid} = \frac{1}{3}V_{prism}$