Noise-induced transition in the Lorenz system

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ICTS (virtual) 8 Dec 2020

Motivation

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What about degenerate noise at high Reynolds number?

Toy model

Edward Lorenz (1963):

$$\dot{X} = \sigma \cdot (Y - X)\,, \qquad \dot{Y} = X \cdot (\varrho - Z) - Y\,, \qquad \dot{Z} = X \cdot Y - \beta \cdot Z\,.$$

Origin globally stable when $\varrho < 1$. Strange attractor for large range of ϱ's.

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Vertical axis $(X = Y = 0)$ always invariant.

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white noise

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Add noise only to Z component.

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Theorem (Coti Zelati, H. '20): When $\varrho < 1$: unique invariant measure for $\alpha \ll 1$, exactly two ergodic I.M.'s for $\alpha \gg 1$.

Change of variables

Equivalent to

$$\dot{x} = y\ , \qquad \dot{y} = -2y - (2 - z)x\ , \qquad \dot{z} = -\gamma(z - z_\star) + \alpha z - x(x + \eta y)\ ,$$

with $z_\star = 2$ iff $\varrho = 1$ and new α proportional to old one. Consider $\gamma > 0$ and $\eta \in \mathbf{R}$ fixed.

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- Overdamped harmonic oscillator for $z \in (1, 2)$.
- Underdamped harmonic oscillator for $z < 1$.
- Unstable for $z > 2$.

Separation of timescales heuristic

For z fixed, eigenvalues of (x, y) system given by

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Guess: vertical axis becomes unstable when

$$\mathrm{Re} \int \lambda_\pm(z) \, \mathcal{N}(z_\star, \alpha^2/2\gamma)(dz) > 0 \, .$$

Not quite true but almost, especially for large α since heuristic good for large z...

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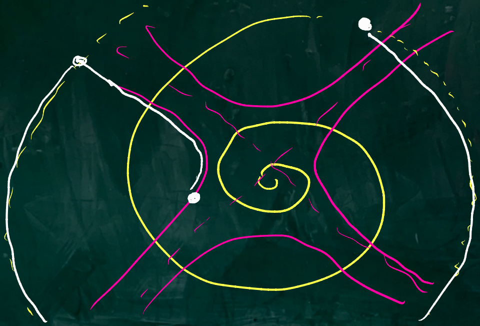
Noise propagates to all directions

Standard criterion: hypoellipticity + controllability. Both quite easy to check.



Another change of variables

Setting $x = e^r \sin\theta$ and $y = e^r(\cos\theta - \sin\theta)$, one has

$$\dot\theta = 1 - z\,\sin^2\theta\;, \qquad \dot r = -1 + \frac{z}{2}\sin 2\theta\;, \qquad \dot z = -\gamma(z - z_\star) + \alpha\xi + (\ldots)\;.$$

Write μ_α for (unique) invariant measure for (θ, z) component with $(\ldots)$ neglected and set $v_\alpha = \int z \sin 2\theta\,\mu_\alpha(d\theta, dz)$.

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Theorem: One has $v_\alpha = 2\sqrt{z_\star - 1}\mathbf{1}_{z_\star > 1} + \mathcal{O}(\alpha^{3/4})$ for $\alpha \ll 1$
$v_\alpha = c\sqrt{\alpha} + \mathcal{O}(\alpha^{1/3})$ (explicit c) for $\alpha \gg 1$.

Lift dynamic to a Markov chain on space E of pieces of trajectories, i.e. $\hat{\mu}_\alpha$

$$F : \mathbb{R}^3 \to \mathbb{R} \quad \Rightarrow \quad \hat{F} : E \to \mathbb{R}$$
$$u \longmapsto \int_0^{\tau(u)} F(u(s))\, ds$$

Prop: $$\int F(u)\, \mu_\alpha(du) = \frac{\int \hat{F}(U)\, \hat{\mu}_\alpha(dU)}{\int \tau(U)\, \hat{\mu}_\alpha(dU)}$$

<u>Idea</u>: For $F(U) = z \sin 2\theta$,

$$\int_0^{\tau(U)} \hat{F}(U_s)\, ds \simeq 2\int_0^{\tau(U)} \sqrt{z_s}\, \mathbb{1}_{z_s > 0}\, ds + O(\dots)$$

$$\Rightarrow \int z \sin 2\theta \, d\mu_\alpha \simeq 2 \int \sqrt{z}\, \mathbb{1}_{z>0}\, d\mu_\alpha$$

↑ known.