Statistical Mechanics of Vortices

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Outline

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1. Moduli space of Abelian Higgs Vortices

- Consider Abelian Higgs (Ginzburg–Landau) vortices on a (closed, oriented, Riemannian) surface Σ of genus g, with 2nd-order dynamics.
- We use a (local) complex coordinate $z = x_1 + ix_2$ and express the metric on Σ as

$$ds^2 = \Omega(z, \bar{z})\, dz\, d\bar{z}\,.$$

- The curvature of Σ affects vortex solutions, but vortices have no back-reaction on the geometry.
- The fields are a section and connection on a $U(1)$ bundle over Σ, with first Chern number $N > 0$. Locally, they are a complex scalar field ϕ and a Maxwell vector potential A_j ($j = 1, 2$), with field strength $F_{12} = \partial_1 A_2 - \partial_2 A_1$ and total magnetic flux $2\pi N$.

- At critical coupling, the minimal energy fields, for given N, satisfy the Bogomolny equations

$$\begin{aligned} D_1\phi + iD_2\phi &= 0\,, \\ \frac{1}{\Omega}F_{12} &= \frac{1}{2}(1-|\phi|^2)\,. \end{aligned}$$

- Provided the surface area satisfies

$$A \equiv \int_\Sigma \Omega\, d^2x > 4\pi N$$

these equations have non-trivial, static N-vortex solutions. The vortex centres are where both $\phi = 0$ and the magnetic field $\frac{1}{\Omega}F_{12}$ is maximal.
- There is a unique solution (up to gauge transformations) with vortex centres at any N specified locations $Z_1, Z_2, \ldots, Z_N$ [Taubes, Bradlow, Garcia-Prada].

- $\mathcal{M}$, the moduli space of solutions, is the symmetrised Nth power Σ^N/S_N, where S_N is the permutation group. $\mathcal{M}$ is a smooth manifold of complex dimension N (not an orbifold).
- $\mathcal{M}$ has a natural metric. This is derived from the field theory kinetic energy, for vortices moving adiabatically through (i.e. tangent to) the moduli space.
- Samols (and Strachan in a special case) showed that the metric has a localised form depending only on data close to each vortex location.
- Slow-motion vortex dynamics corresponds to free motion (i.e. geodesic motion) on the moduli space $\mathcal{M}$. The vortices themselves have non-trivial interactions because of the curvature of $\mathcal{M}$. We ignore dynamics orthogonal to $\mathcal{M}$ that excites vortices.

2. Classical Partition Function and Eq. of State

- Here we use real coordinates $\{q^i : 1 \leq i \leq 2N\}$ on the moduli space $\mathcal{M}$ (real and imaginary parts of Z_r), and denote the Samols metric $g_{ij}(\mathbf{q})$.
- The Hamiltonian for free motion on $\mathcal{M}$ is

$$H(\mathbf{p}, \mathbf{q}) = \frac{1}{2} g^{ij}(\mathbf{q}) p_i p_j \,,$$

where $p_i = g_{ij}(\mathbf{q})\dot{q}^j$ are the momenta conjugate to q^i.
- The classical partition function at temperature T is

$$Z(T) = \frac{1}{(2\pi\hbar)^{2N}} \int e^{-H(\mathbf{p},\mathbf{q})/T} \, [dp\, dq] \,.$$

- After performing the Gaussian momentum integrals,

$$Z(T) = \left(\frac{T}{2\pi\hbar^2}\right)^N \int_{\mathcal{M}} \sqrt{\det g_{ij}} \, [dq] = \left(\frac{T}{2\pi\hbar^2}\right)^N \mathrm{Vol}(\mathcal{M}) \,,$$

a multiple of the volume of $\mathcal{M}$.

- For N vortices on a surface Σ of genus 0 (topologically, a 2-sphere) with area $A > 4\pi N$, the volume of the moduli space is

$$\mathrm{Vol}(\mathcal{M}) = \frac{\pi^N}{N!}(A - 4\pi N)^N$$

(discussed in Martin Speight's lectures).
- The partition function of the vortex gas is therefore

$$Z(T) = \left(\frac{T}{2\hbar^2}\right)^N \frac{1}{N!}(A - 4\pi N)^N .$$

- This formula has a known generalisation for vortices on a surface of genus g [NSM and S. Nasir], but this makes no difference if $g \ll N$.

- From the partition function Z we obtain the free energy $F = -T \log Z$, where, for large N,

$$F = -T\left\{N\log\left(\frac{T}{2\hbar^2}\right) - N\log N + N + N\log(A - 4\pi N)\right\}.$$

- The pressure of the vortex gas is therefore

$$P = -\frac{\partial F}{\partial A} = \frac{NT}{A - 4\pi N}.$$

  Equivalently

$$P(A - 4\pi N) = NT,$$

  the Clausius equation of state.

- Remarkably, this is an exact result for an interacting system.

3. Quantized Vortices and Eq. of State at High T

- The (canonically) quantized momentum operators on $\mathcal{M}$,

$$p_i = -i\hbar \frac{\partial}{\partial q^i} \,,$$

  are inserted into H to determine the quantum Hamiltonian.
- After operator-ordering,

$$H = \frac{1}{2}\hbar^2 \Delta \,,$$

  where $\Delta \equiv -\nabla^2$ is the Laplace–Beltrami operator on the moduli space $\mathcal{M}$. This is our quantized Hamiltonian for vortices. N and A are both large, with $A > 4\pi N$.
- Since $\mathcal{M}$ is compact, without boundary, Δ has a discrete set of non-negative eigenvalues λ. The quantum partition function is

$$Z(T) = \sum_\lambda \exp\left(-\frac{\hbar^2}{2T}\lambda\right) \,.$$

- It is easiest to find $Z(T)$ at large T.
- The first two terms of the large T expansion are [Berger, Gaudichon and Mazet]

$$Z(T) \sim \left(\frac{T}{2\pi\hbar^2}\right)^N \left(\mathrm{Vol}(\mathcal{M}) + \frac{\hbar^2}{12T}\mathrm{Curv}(\mathcal{M})\right) ,$$

where $\mathrm{Vol}(\mathcal{M})$ is the volume of $\mathcal{M}$ and

$$\mathrm{Curv}(\mathcal{M}) = \int_{\mathcal{M}} s\,\sqrt{\det g_{ij}}\,[dq]$$

its total scalar curvature. Here, s is the local scalar curvature of the Samols metric g_{ij}.

- For N vortices on a surface Σ of genus 0, the total volume is

$$\mathrm{Vol}(\mathcal{M}) = \frac{\pi^N}{N!}(A - 4\pi N)^N$$

and the total curvature is [Baptista]

$$\mathrm{Curv}(\mathcal{M}) = \frac{4N^2\pi^N}{N!}(A - 4\pi N)^{N-1}\,.$$

The two-term partition function is therefore

$$Z(T) = \left(\frac{T}{2\hbar^2}\right)^N \frac{1}{N!}(A - 4\pi N)^N \left[1 + \frac{\hbar^2 N^2}{3T(A - 4\pi N)}\right]\,,$$

the classical result plus the leading quantum correction.

- The free energy $F = -T \log Z$ is now

$$F = -T\left\{N \log\left(\frac{T}{2\hbar^2}\right) - N\log N + N + N\log(A - 4\pi N) + \frac{\hbar^2 N^2}{3T(A-4\pi N)}\right\}.$$

- The pressure of the vortex gas is therefore

$$P = -\frac{\partial F}{\partial A} = \frac{NT}{A - 4\pi N} - \frac{\hbar^2 N^2}{3(A-4\pi N)^2},$$

which can be arranged into the equation of state

$$\left(P + \frac{\hbar^2 N^2}{3(A-4\pi N)^2}\right)(A - 4\pi N) = NT\,.$$

This is similar to the van der Waals equation.

- At low density,

$$P = \frac{NT}{A}\left(1 + \left(4\pi - \frac{\hbar^2}{3T}\right)\frac{N}{A} + \cdots\right)$$

so the 2nd virial coefficient is

$$B(T) = 4\pi - \frac{\hbar^2}{3T}\,.$$

- The vortex gas is repulsive at high temperature, because each vortex excludes some area from the others. The repulsion softens as T decreases, but we cannot extrapolate this result to T of order $\hbar^2$.

4. Statistical Mechanics of Dissolving Vortices [NSM and Shiyi (Franklin) Wang]

- The quantum mechanical spectrum on the N-vortex moduli space is not known in general, but it simplifies near the Bradlow limit, where A/N slightly exceeds 4π.
- The vortices are called "dissolving" in this regime, as the scalar field ϕ is close to zero everywhere, and the magnetic field is nearly uniform.
- The moduli space is CP^N for N vortices on a surface of genus 0, and for dissolving vortices its metric becomes the standard Fubini–Study metric, scaled by $A - 4\pi N$ to have the correct volume [Baptista and NSM, Speight].

- The quantum Hamiltonian becomes

$$H = \frac{1}{2}\hbar^2 \frac{1}{A - 4\pi N} \Delta_{\mathrm{FS}} \,.$$

- Δ_{FS} has eigenvalues and degeneracies

$$\lambda_k = 4k(N+k)\,, \quad g_k = \binom{N+k}{k}^2 - \binom{N+k-1}{k-1}^2 ,$$

with $k = 0, 1, 2, \ldots$.

- The quantum partition function is therefore

$$Z(T) = \sum_k g_k \exp\left(-\frac{\hbar^2}{2T}\frac{4k(N+k)}{A - 4\pi N}\right) .$$

- For all except very low temperatures, this sum is dominated by a range of k of order N. So can replace sum by integral.
- Introduce $x = \frac{k}{N}$ and the scaled reciprocal temperature

$$z = \frac{\hbar^2}{2\pi T} \frac{4\pi N}{A - 4\pi N} \,.$$

- Find

$$Z(z) \simeq N \int_0^\infty \exp(NG(x))\, dx$$

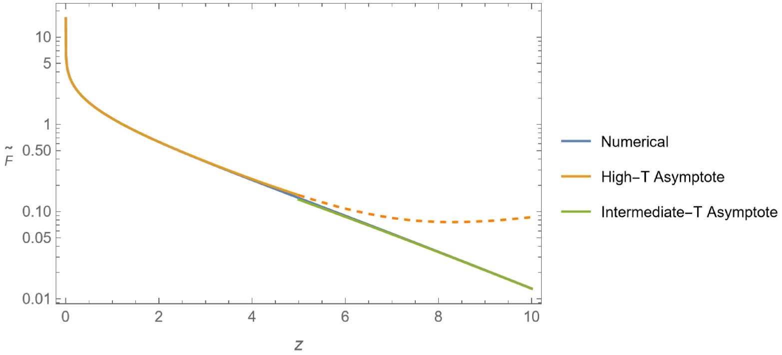
where

$$G(x) = 2(1+x)\log(1+x) - 2x\log x - z\, x(1+x)\,.$$

- As N is large, we approximate integrand by a Gaussian around maximum of $G(x)$, whose location is where

$$\log\left(1+\frac{1}{x}\right) = z\left(x+\frac{1}{2}\right) .$$

- We have found the solution $x_0(z)$ numerically, and have also found asymptotic formulae for $x_0(z)$ for small z (high T) and large z (intermediate to low T).
- We can then calculate the partition function Z, the free energy $F = -T\log Z = -NTG(x_0(z))$, and pressure $P = -\partial F/\partial A$ of the vortex gas. (Recall $z = \frac{\hbar^2}{2\pi T}\frac{4\pi N}{A-4\pi N}$.)



Scaled free energy $F/(-NT)$ for small and large z
($z \ll 2 \log N$)

- Pressure for small and large z are

$$P = \frac{NT}{A-4\pi N}\left[1-\frac{z}{6}+\frac{z^2}{180}+\dots\right] \qquad (\text{small } z)\,,$$
$$P = \frac{NT}{A-4\pi N}\left[ze^{-\frac{1}{2}z}+z(2-z)e^{-z}\right] \qquad (\text{large } z)\,.$$

  The first two terms for small z (high T) reproduce the classical pressure and first quantum correction.
- Replacing sum by Gaussian integral fails for $z > 2\log N$ (very low T). But sum simplifies here to a modified Bessel function series, so

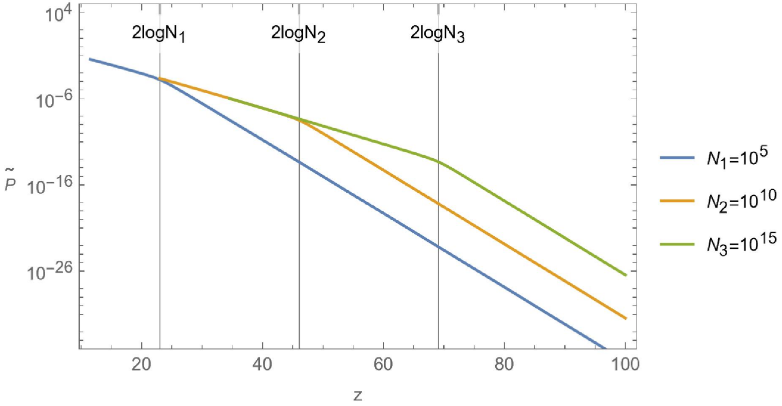
$$Z(z) \simeq I_0(2Ne^{-\frac{1}{2}z}) \simeq 1 + N^2 e^{-z}\,.$$

- The free energy at very low T is therefore

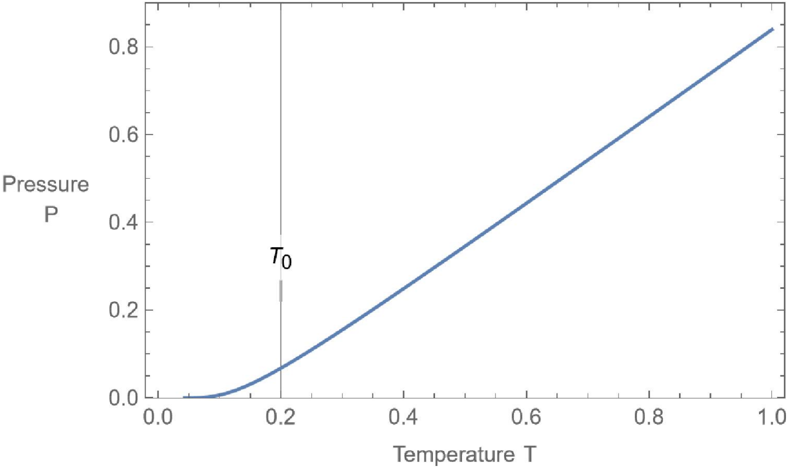
$$F = -T\log Z \simeq -TN^2 e^{-z}\,.$$

Curiously, F is not extensive (proportional to N), but proportional to N^2. Pressure P proportional to N at fixed density, and exponentially small.

- These results suggest that quantum mechanical vortices behave like bosons in this model.
- Evidence: (a) Leading high-T quantum correction reduces the classical pressure; (b) pressure is very small at very low T.



Scaled pressure for large and very large z



Scaled pressure against scaled temperature; T_0 corresponds to $z = 5$

5. Conclusions

- Classical statistical mechanics of (critically-coupled) Abelian Higgs vortices exactly solvable, despite vortex interactions.
- Classical equation of state has been extended to include first quantum correction at high temperature T.
- Calculations use exact results for volume and total scalar curvature of N-vortex moduli space.
- For dissolving vortices, moduli space metric simplifies to Fubini–Study. Exact quantum energy spectrum known, so partition function calculated for all T. Asymptotic formulae found for pressure of vortex gas at high, intermediate to low, and very low T.