

Recall: $L_p^{(r)}(x, \omega, 0) = r! L_R(x, 0) R_p(x)$.

Lecture 3 Conjecture 1 $\iff R_p(x) \neq 0$ (This part was conjectured by Kuzmin)

Remark: This statement is independent of conjecture 1. If conj 1 fails we show $R_p(x) = 0$.

§ Reformulation in terms of existence of Galois cohomology classes.

$$R_1 = \{ \sigma \in G_F(x) \mid \pi(\sigma) = 1 \}$$

$$H'_{R_1}(G_F, E(x^{-1})) \subseteq H'(G_F, E(x^{-1}))$$

E with G_F action by x^{-1}
 $\sigma \cdot e = x^{-1}(\sigma)e$

subspace of classes unramified outside primes in R_1 i.e.

$$H'_{R_1}(G_F, E(x^{-1})) = \ker \left(H'(G_F, E(x^{-1})) \xrightarrow{\text{res}} \prod_{v \notin R_1} H'(I_v, E(x^{-1})) \right)$$

$\uparrow$ inertia subgroup

As $\chi(G_f) = 1$ for $f \in R_1$ hat means p-adic completion

$$H^1(G_f, E(X^{-1})) = \text{Hom}_{\text{cts}}(G_f, E) \\ = \text{Hom}_{\text{cts}}(\widehat{F}_f^{\times}, E).$$

decomposition subgroup.

Local class field theory

There are two homomorphisms in this

$$\begin{array}{ccc} l_f, & o_f : & \widehat{F}_f^{\times} \longrightarrow \mathbb{Z}_f \subseteq E \\ \text{"log Norm}_{F_f/\mathbb{Q}_p} & \text{"ord}_f & \end{array}$$

E vector space generated by l_f, o_f

$$H^1_{\text{cyc}}(X) \subseteq H^1_{R_1}(G_F, E(X^{-1}))$$

$$\left\{ \kappa \in H^1_{R_1}(G_F, E(X^{-1})) \mid \text{res}_f(\kappa) \in \langle l_f, o_f \rangle \right\} \\ \forall f \in R_1$$

Proposition: $\dim_E H'_{\text{cyc}}(X) = r$ (Dasgupta - Darmon Pollack)

Proof: Poincaré-Tate sequence gives Tate twist.

$$0 \rightarrow H'_{\text{cyc}}(X) \rightarrow \bigoplus_{\beta \in R_1} \langle \phi_\beta, \ell_\beta : \beta \in R_1 \rangle \rightarrow H'_{R_1}(G_F, E(1)(X))^v \rightarrow 0$$

$\uparrow$ $\dim 2r$ $\uparrow$ $\dim r$

$(V \otimes E)^X$

□

Notation: $R_1 = \{\beta_1, \dots, \beta_r\}$

res_{β_j} $\kappa_1, \dots, \kappa_r$ an E -basis for $H'_{\text{cyc}}(X)$

Notation: $\text{res}_j(\kappa_i) = x_{ij} \phi_j + y_{ij} \ell_j$
 $(x_{ij})_{1 \leq i \leq r, 1 \leq j \leq r}$
 $(y_{ij})_{1 \leq i \leq r, 1 \leq j \leq r}$

Proposition: $R_p(X) = \det(x_{ij}) / \det(y_{ij})$. (Greenberg - Stevens)

Proof: Let $u_1, \dots, u_r$ be an E -basis of $(E \otimes U)^X$.

Poitou-Tate duality $\rightarrow \sum_{j=1}^r (\text{res}_j(u)) (u) = 0 \quad \forall u \in \hat{F}_{\partial_j}^X \otimes E$
 for any $\kappa \in H_{\mathbb{Z}}^1$

$$0 = \sum_{j=1}^r \text{res}_j(\kappa_i)(u_k) = \sum_j x_{ij} o_j(u_k) + \sum_j y_{ij} l_j(u_k).$$

$$\det[(x_{ij})_{i,j} \quad (o_j(u_k))_{j,k}] = \det(-(y_{ij}) \cdot (l_j(u_k))_{j,k}). \quad \square$$

Upside: It is enough to construct $\kappa_1, \dots, \kappa_r$
 s.t. $\det(x_{ij}) / \det(y_{ij}) = \frac{L_p^{(r)}(X, 0)}{r! \cdot L_R(X, 0)}$

Remark This is exactly what Dasgupta-Dannan-Pollack do (+ Ventullo) when $r=1$.

When $r > 1$ we do not know how to construct

even a single class in $H_{\text{cyc}}^1(X)$.

§ Construction of a cusp form

$E(1, X)$
looks cuspidal
modulo
 $L_p(X_w, s)$.

Recall that we have a Λ -adic Eisenstein series

$E(1, X)$ with "constant terms" $L_p(X_w, s) \in \Lambda$.

Question: How do we get a cusp form from this.

(Ribet, Wiles) Construct a Λ -adic modular form g with constant term 1.

Then $E(1, X) - L_p(X_w, s)g$ is cuspidal at the cusp at infinity.

Remark: g is not explicitly known. It may not even be an eigenform. Therefore we know

action of Hecke operators on $\{ (1, \chi) - L_p(\chi \omega, s) \}_s$
 only modulo $L_p(\chi \omega, s)$. Furthermore, we do
 not see $L_p(\chi, 0)$ here.

An important idea in Dasgupta-Darmon-Pollack

A refined construction of cusp form.

$$\tilde{f} = \{ (1, \chi) - \frac{L_p(\chi \omega, s)}{L_p(\chi, 0)} E_1(1, \chi_R) \}_s.$$

in T^r
mod T

f - "Lift of Hasse invariant". constant term

1 and non-constant terms are all
 divisible by T

$$\lambda = O_E[T]$$

Proposition (Hida) $\exists$ a Λ -adic form g s.t.

$$v_k(g) \in M_{k-1}(p, \omega^{k-1}), \quad c_\lambda(0, g) = 1 \quad \forall \lambda,$$

and $v_1(g) = 1.$ $g = \tilde{g}(T(u+1)^{-1} - 1)$

Define: Case 1. Assume $R \neq \emptyset$

$$\tilde{f} = E(1, X) - \frac{L_p(X, s)}{L_R(X, 0)} E_1(1, X) g.$$

$$v_k(\tilde{f}) = \tilde{F}_k = E_k(1, X) - \frac{L_p(X, 1-k)}{L_R(X, 0)} E_1(1, X) G_{k-1}.$$

$\tilde{f}$ is cuspidal at ∞ .

Fact: $\tilde{f} = \text{cusp} + \sum_{(\eta, \psi)} a(\eta, \psi) E(\eta, \psi)$

$\exists$ a Hecke operator t s.t. $t \cdot e \cdot \tilde{f}$ is a cusp form.

$\uparrow$
 Hida's ordinary operator $e = \lim_{n \rightarrow \infty} U_p^{n!}$

Case 2: $R = \emptyset$. In this case there is $\xi(x, 1)$ which is also ordinary & $v_1(\xi(x, 1)) = E_1(x, 1) = E_1(1, x)$.

$$f = t \cdot e \cdot \left(\xi(1, x) - \frac{L_p(xw, s)}{L(x, 0)} E_1(1, x) \cdot e_y + \frac{L_p(xw, s)}{L_p(x^{-1}w, s)} \cdot \frac{L(x^{-1}, 0)}{L(x, 0)} \xi(x, 1) \right)$$

↑
(not necessarily an) cusp form

is cuspidal.

If $R \neq \emptyset$
 $v_1(\xi(x, 1)) = E_1(x, 1)$
 ↑
 not ordinary
 $e \cdot E_1(x, 1) = 0$

Change of variable: $\Lambda = \mathcal{O}_E[[T]]$ $\pi = \frac{1}{\log_p u} \cdot T$.

Π = cuspidal Λ -adic Hecke algebra, generated by $T_\ell, U_q, q \mid \pi p$. ($\pi = \text{cond}(x)$)

$$r_{an} = \text{ord}_{s=0} L_p(x_{\omega}, s)$$

$$L_{an}^*(x) = L_p^{(r_{an})}(x_{\omega}, 0) / r_{an}! L_R(x, 0)$$

Theorem: Suppose $R \neq \emptyset$. Then there exists a Λ -alg homomorphism $\varphi: \pi \longrightarrow W$, surjective,

where $W = E[\pi, \varepsilon_1, \dots, \varepsilon_r] / (\pi^{r_{an}+1}, \varepsilon_i^2, \varepsilon_i \pi, \varepsilon_1 \dots \varepsilon_r + (-1)^r L_{an}^*(x) \pi^{r_{an}})$

such that

$$T_l \mapsto 1 + \chi \varepsilon(l)$$

$$\forall l \in \pi \quad \text{\textit{\(\Lambda\}-adic cyclotomic character}}$$

$$U_l \mapsto 1$$

$$\forall l \in R \text{ or } l \in \pi$$

$$U_{f_i} \mapsto 1 + \varepsilon_i$$

$$R_1 = \{f_1, \dots, f_r\}$$

Proof:
 $g = t \cdot e \cdot f$

$$\tilde{f} = \xi(1, x) - \frac{L_p(x_{\omega}, s)}{L_R(x, 0)} E_1(1, x) g.$$

and constant terms of g are 1
non-constant terms of g are in (π) .

$$\Rightarrow \left(\begin{aligned} &U_{\beta_i} E_1(1, \chi_R) + E_1(1, \chi_{R \cup \{\beta_i\}}) \end{aligned} \right) \tilde{F} \equiv \mathcal{E}(1, \chi) - \mathcal{L}_{an}^*(\chi) E_1(1, \chi_R) \pi^{r_{an}} \pmod{\pi^{r_{an}+1}}$$

$$\begin{aligned} &= E_1(1, \chi_R) + E_1(1, \chi_{R \cup \{\beta_i\}}) \\ &U_{\beta_i} \tilde{F} \equiv \mathcal{E}(1, \chi) - \mathcal{L}_{an}^*(\chi) E_1(1, \chi_R) \pi^{r_{an}} \\ &\quad \xrightarrow{U_{\beta_i} \text{ acts as } 1} - \mathcal{L}_{an}^*(\chi) E_1(1, \chi_{R \cup \{\beta_i\}}) \pi^{r_{an}}. \end{aligned}$$

$$(U_{\beta_i} - 1) \tilde{F} \equiv - \mathcal{L}_{an}^*(\chi) E_1(1, \chi_{R \cup \{\beta_i\}}) \pi^{r_{an}}$$

$$\begin{aligned} \prod_{i=1}^r (U_{\beta_i} - 1) \tilde{F} &\equiv (-1)^r \mathcal{L}_{an}^*(\chi) E_1(1, \chi_S) \pi^{r_{an}} \\ &\equiv (-1)^r \mathcal{L}_{an}^*(\chi) \pi^{r_{an}} \tilde{F} \end{aligned}$$

