Standard compact Clifford-Klein forms and Lie algebra decompositions

joint work with Aleksy Tralle

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ZARISKI DENSE SUBGROUPS, NUMBER THEORY AND GEOMETRIC APPLICATIONS

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1. Proper actions
   - Definition of a proper action
   - Clifford-Klein forms
2. Kobayashi's criterion for proper actions and standard compact Clifford-Klein forms
3. Structural constrains on $(\mathfrak{g}, \mathfrak{h}, \mathfrak{l})$
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Definition

An action of a topological (Hausdorff) group G on a locally compact Hausdorff space M is called proper if $G_S = \{g \in G \mid gS \cap S \neq \emptyset\}$ is compact for every compact $S \subset M$.

Equivalently an action of G on M is proper if the map

$$G \times M \to M \times M, \ (g, m) \mapsto (m, gm)$$

is a proper map (that is, a pre-image of any compact set is compact).

Example

- The action of a Lie group G on itself (by left multiplication) is proper.
- The conjugation action of a non-compact Lie group on itself is not proper.

Let G be a (semisimple) Lie group, $H \subset G$ a closed (reductive) connected subgroup and $\Gamma \subset G$ a discrete subgroup.

Definition

The space $\Gamma \backslash G/H$ is called a Clifford-Klein form if Γ acts properly and freely on G/H. If $\Gamma \backslash G/H$ is compact then it is called a compact Clifford-Klein form.

We also say that Γ is a Clifford-Klein form for G/H and that G/H admits a compact Clifford-Klein form. If $\Gamma \backslash G/H$ is compact then we also say that G/H admits a tessellation. Notice that the assumption that Γ acts freely on G/H is not very significant.

Questions:

Q1 When does a "large" discrete subgroup of G act on a homogeneous space G/H properly?

Q2 When does the homogeneous space G/H admit a compact Clifford-Klein form?

(M, J)- a smooth manifold with a geometric structure J,

$\tilde{M}$- the universal covering of M

$p : \tilde{M} \to M$, $\tilde{o} \in \tilde{M}$

$G := Aut(\tilde{M}, J)$, $H := \{g \in G \mid g\tilde{o} = \tilde{o}\}$ $\Gamma := \pi_1(M, o)$ $(o := p(\tilde{o}))$

In this case $\Gamma \subset G$ and

Theorem

Assume that G is a Lie group and acts transitively on $\tilde{M}$. Then M is diffeomorphic to $\Gamma \backslash G / H$.

When (M, J) is a pseudo-Riemannian manifold then the group G is a Lie group. If it acts transitively on M then $M \cong \Gamma\backslash G/H$.

A *space form* is a pseudo-Riemannian manifold M with constant sectional curvature. A complete space form of signature (p, q), $p \geq 2$ with a constant positive sectional curvature κ is a Clifford-Klein form of the symmetric space

$$O(p+1, q)/O(p, q)$$

Theorem (Kulkarni)

If p, q are odd then there is no compact Clifford-Klein form of $O(p+1, q)/O(p, q)$.
There exist compact complete pseudo-Riemannian space forms of signature $(3, 4n)$.

Idea of the proof: find a connected group (with a co-compact lattice) in $SO(4, 4n)$ that acts properly and co-compactly on $O(4, 4n)/O(3, 4n)$.

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Let

- G be a connected linear semisimple real Lie group with the Lie algebra $\mathfrak{g}$.
- H, L be connected reductive subgroups of G,
- θ be a Cartan involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition w.r.t. θ.

We may assume that $\theta|_h$ and $\theta|_l$ are Cartan involutions of $\mathfrak{h}$, $\mathfrak{l}$, respectively. Thus we have compatible decompositions

$$\mathfrak{h} = \mathfrak{k}_h + \mathfrak{p}_h, \quad \mathfrak{l} = \mathfrak{k}_l + \mathfrak{p}_l.$$

Choose a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$.

Definition

The real rank of a Lie algebra $\mathfrak{g}$ (denoted $\operatorname{rank}_{\mathbb{R}} \mathfrak{g}$) is the dimension of $\mathfrak{a}$.

Denote by K the maximal compact subgroup of G with the Lie algebra $\mathfrak{k}$. Let $W := N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ be the Weyl group of G.

Denote by $\mathfrak{a}_h, \mathfrak{a}_l$ the maximal abelian subspaces of $\mathfrak{p}_h$ and $\mathfrak{p}_l$, respectively. We may assume that $\mathfrak{a}_h, \mathfrak{a}_l \subset \mathfrak{a}$.

Theorem (Kobayashi)

The following conditions are equivalent
(i) H acts on G/L properly,
(ii) L acts on G/H properly,
(iii) $\mathfrak{a}_h \cap W\mathfrak{a}_l = \{0\}$.
Moreover, the subgroup L acts properly on G/H only if

$$\mathrm{rank}_{\mathbb{R}}(\mathfrak{l}) + \mathrm{rank}_{\mathbb{R}}(\mathfrak{h}) \leq \mathrm{rank}_{\mathbb{R}}(\mathfrak{g}).$$

Example

The one sheeted hyperboloid $SL(2,\mathbb{R})/SO(1,1)$ admits only finite Clifford-Klein forms.

Idea: find a reductive subgroup L of G that acts properly and co-compactly on G/H and take a co-compact lattice $\Gamma \subset L$. Kobayashi gave the following list of triples $(\mathfrak{g}, \mathfrak{h}, \mathfrak{l})$ of Lie algebras generating compact Clifford-Klein forms:

$$(\mathfrak{su}(2,2n), \mathfrak{sp}(1,n), \mathfrak{u}(1,2n))$$

$$(\mathfrak{so}(2,2n), \mathfrak{so}(1,2n), \mathfrak{u}(1,n))$$

$$(\mathfrak{so}(4,4n), \mathfrak{so}(3,4n), \mathfrak{sp}(1,n))$$

$$(\mathfrak{so}(4,3), \mathfrak{so}(4,1), \mathfrak{g}_2)$$

$$(\mathfrak{so}(8,8), \mathfrak{so}(7,8), \mathfrak{so}(1,8))$$

Surprisingly, these are the only known examples of homogeneous spaces admitting compact Clifford-Klein forms (with $\mathfrak{g}$ real, absolutely simple and $\mathfrak{h}$ simple of non-compact type).

Described examples can be obtain in the following way. Assume that

$$G = HL \text{ and } H \cap L \text{ is compact.}$$

Then

$$G/H \cong L/(H \cap L)$$

so L (and any closed subgroup of L) acts properly on G/H. Therefore any co-compact lattice of L is a compact Clifford-Klein form of G/H.

Definition

A compact Clifford-Klein such that $\Gamma \subset L$ is called *standard.*

Classification of all triples $(\mathfrak{g}, \mathfrak{h}, \mathfrak{l})$, $\mathfrak{g}$-simple, $\mathfrak{h}, \mathfrak{l}$-semisimple, such that $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$ was obtained by Onishchik (9).

Conjecture (Kobayashi)

If G/H admits a compact Clifford-Klein form then it admits a standard compact Clifford-Klein form.

But not all compact Clifford-Klein forms are standard:

Kassel, Kobayashi, Salein: Deformations of standard Clifford-Klein forms (e.g. $SO(2,2n)/SO(1,2n)$).

Monclair, Schlenker, Tholozan: Exotic compact Clifford-Klein forms of $O(2,2n)/U(1,n)$.

Problem: classify standard Clifford-Klein forms.

In recent years the following results concerning standard compact Clifford-Klein forms were obtained.

Theorem (Tojo (11))

Let G/H be a non-compact irreducible simple symmetric space which admits a standard compact Clifford-Klein form. Then $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$.

Theorem ((1))

Let G/H be a non-compact reductive homogeneous space of a real linear simple exceptional Lie group G. Then G/H admits a standard compact Clifford-Klein form if and only if H is compact.

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Assume that $(\mathfrak{g}, \mathfrak{h}, \mathfrak{l})$ corresponds to a standard compact Clifford-Klein form (i.e. L acts properly and co-compactly on G/H). Moreover assume that G is absolutely simple and non-compact, H, L are reductive.

Idea

- If $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$ then $(\mathfrak{g}, \mathfrak{h}, \mathfrak{l})$ is contained in (9) .
- If $\mathfrak{g} \neq \mathfrak{h} + \mathfrak{l}$ then there exists $0 \neq X \in \mathfrak{g}$ which is orthogonal (w.r.t. the Killing form of $\mathfrak{g}$) to $\mathfrak{h} + \mathfrak{l}$. Using X one can obtain structural restrictions on $\mathfrak{h}$ and $\mathfrak{l}$.

Recall that $\mathfrak{a} \subset \mathfrak{p}$ denotes a maximal abelian subspace and let $\mathfrak{m}_0$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Choose a maximal Cartan subalgebra $\mathfrak{t}$ in $\mathfrak{m}_0$. Let $\Sigma \subset \mathfrak{a}^*$ be the real root system of $\mathfrak{g}$ determined by $\mathfrak{a}$. The subalgebra $\mathfrak{j}^C = (\mathfrak{t} + \mathfrak{a})^C$ is a Cartan subalgebra of $\mathfrak{g}^C$ so we can take the corresponding root system Δ of $\mathfrak{g}^C$. Without loss of generality we assume that $\mathfrak{a}_h, \mathfrak{a}_l \subset \mathfrak{a}$ (denote by Σ_h, Σ_l be the real root systems of $\mathfrak{h}$ and $\mathfrak{l}$, respectively). For any $\gamma \in \Sigma$ we denote by $\mathfrak{g}_\gamma$ the corresponding root space and by $\mathfrak{g}_\gamma^C$ its complexification.

Let

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}, \quad \mathfrak{h} = \mathfrak{k}_h + \mathfrak{a}_h + \mathfrak{n}_h, \quad \mathfrak{l} = \mathfrak{k}_l + \mathfrak{a}_l + \mathfrak{n}_l.$$

be the compatible Iwasawa decompositions.

Assume that $0 \neq X \in \mathfrak{t}$. Define

$$\Delta_m = \{\alpha_c \in \Delta \mid \alpha_c|_{\mathfrak{a}} = 0\},$$

$$\Delta_m^+ = \Delta^+ \setminus \Delta_m, \Delta_m^- = \Delta^- \setminus \Delta_m,$$

$$\Delta_p = \{\alpha_c \in \Delta_m^+ \mid \alpha_c(iX) > 0\}, \; \Delta_n = \{\alpha_c \in \Delta_m^+ \mid \alpha_c(iX) < 0\},$$

$$\Delta_p^- = \{\alpha_c \in \Delta_m^- \mid \alpha_c(iX) > 0\}, \; \Delta_n^- = \{\alpha_c \in \Delta_m^- \mid \alpha_c(iX) < 0\},$$

$$\Delta_0 = \{\alpha_c \in \Delta_m^+ \mid \alpha_c(iX) = 0\}, \; \Delta_0^- = \{\alpha_c \in \Delta_m^- \mid \alpha_c(iX) = 0\}.$$

Proposition

Assume that (G, H', L') is a standard triple. Then there is an equivalent standard triple (G, H, L) (H, L-semisimple) such that

$$\mathfrak{n} = \mathfrak{n}_h \oplus \mathfrak{n}_l.$$

Let

$$Z = \sum_{\alpha_c \in \Delta_p \cup \Delta_n} \mathfrak{g}_{\alpha_c} \subset \mathfrak{n}^C.$$

Let $\pi : \mathfrak{n}^C = \sum_{\alpha_c \in \Delta_m^+} \mathfrak{g}_{\alpha_c} \to Z$ be the natural projection, put

$$Z_h = \pi(\mathfrak{n}_h^C), \quad Z_l = \pi(\mathfrak{n}_l^C).$$

Theorem

Assume that $\mathfrak{g} \neq \mathfrak{h} + \mathfrak{l}$ *so there exists a non-zero* $X \in \mathfrak{t}$ *such that* X *is orthogonal to* $\mathfrak{h} + \mathfrak{l}$. *We have the following:*

1. *There exists a basis of* Z_h *of the form*

$$S_h^i = x_{\alpha_i} + \sum_{l=1}^{k} a_{k+l}^i x_{\alpha_{k+l}}, a_{k+l}^i \in \mathbb{C}, \alpha_i \in \Delta_p, \alpha_{k+l} \in \Delta_n.$$

2. *For any* $S_h^i \in Z_h$ $\alpha_i|_{\mathfrak{a}_h} \in \Sigma_h$.
3. *Each complexified real root space* $\mathfrak{h}_\gamma^c$ *is spanned by vectors of the form*

$$S_h^{i_1} + Q_1, \ldots, S_h^{i_s} + Q_s, Q_{s+1}, \ldots, Q_{s+w}, s + w = \dim h_\gamma^c,$$

where $\alpha_{i_1}, \ldots, \alpha_{i_s}$ *are all roots from* Δ_p *whose restrictions onto* $\mathfrak{a}_h$ *coincide with* γ, *while all* Q_j *satisfy the conditions*

$$Q_j \in \sum_{\alpha_c \in \Delta_0, \ \alpha_c|_{\mathfrak{a}_h} = \gamma} \mathfrak{g}_{\alpha_c}.$$

We say that $\mathfrak{h}$ is a regular subalgebra of $\mathfrak{g}$ if $\mathfrak{a}$ normalizes $\mathfrak{h}$. We say that $\mathfrak{h}$ is a proper regular subalgebra of $\mathfrak{g}$ if $\mathrm{rank}_{\mathbb{R}}(\mathfrak{h}) < \mathrm{rank}_{\mathbb{R}}(\mathfrak{g})$. Analogously to Dynkin we say that a real semisimple subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is a (proper) real R-regular subalgebra, if it is contained in a (proper) regular subalgebra (notice that all semisimple Lie subalgebras in a simple complex Lie algebra also fall into two classes: R-regular and S-subalgebras, according to Dynkin). The previous theorem fully settles the case of real R-regular subalgebras.

Theorem

If $\mathfrak{h}$ is a proper real R-regular subalgebra in $\mathfrak{g}$ (of non-compact type), then no G/H admits a compact Clifford-Klein form.

Theorem

If $\mathfrak{g}$ is a simple split Lie algebra, then $(\mathfrak{g}, \mathfrak{h})$ determines a standard compact Clifford-Klein form if and only if there exists a semisimple Lie algebra $\mathfrak{l} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$.

Corollary

Let G be a linear connected Lie group whose Lie algebra is one of the following

$$\mathfrak{sl}(n,\mathbb{R}),\ n>1, \quad \mathfrak{so}(n,n),\ n=3,5,7,n>9,$$

$$\mathfrak{so}(n,n+1),\ n=2,n>3, \quad \mathfrak{sp}(n,\mathbb{R}),\ n>1.$$

Choose any reductive subgroup $H\subset G$ such that G/H is non-compact. Then G/H admits a standard compact Clifford-Klein form if and only if H is compact.

Corollary

Let G be a linear connected Lie group whose Lie algebra is $\mathfrak{su}(n,m)$, $m\geq n>2$, ($\mathfrak{so}(n,m)$, $m+1>n>8$). Let $H\subset G$ be a reductive subgroup whose Lie algebra is a subalgebra of a proper regular subalgebra $\mathfrak{su}(n-1,m-1)$ ($\mathfrak{so}(n-1,m-1)$) such that G/H is non-compact. Then G/H admits a standard compact Clifford-Klein form if and only if H is compact.

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