Maths Circle Explorations: Session 6

TIFR, Mumbai

7th January 2022

Problem 1

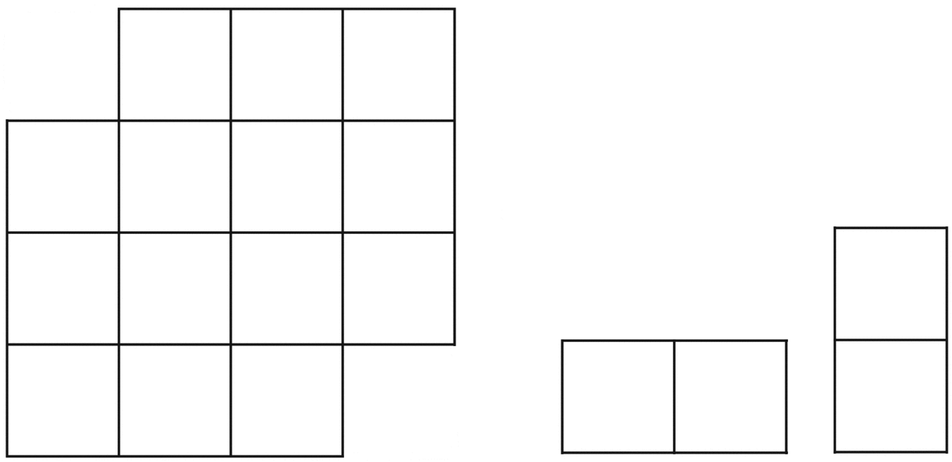
Suppose there is a 1000 doors in a row marked with numbers from $1,..,1000$. All the doors are white. You are asked to paint the doors any way you want, but each door should be single-coloured (like upper half of a door is blue and the lower half is yellow, is invalid colouring). Being a mathematics lover, you decide to paint the doors in a pattern, but also you don't want a bystander to just look at the doors and recognise your pattern. So, you device a strategy,
You choose 2 primes, $2,3$.
You paint the doors with numbers in the sequence $1,3,5,7,9,11,13,.....$ (numbers of the form $1+2k$), with red colour.
You paint the doors with numbers in the sequence $1,4,7,10,13,16,.....$(numbers of the form $1+3k$) with blue colour.
So, there were some white doors, some red doors, some blue doors, and some purple doors (red+blue=purple).
Then you called your friend, who is also a mathematics lover, to show your work. As it turned out, he easily found out exactly what is the procedure you followed (only he said he didn't know whether you coloured blue first or red first).
So you send him back to get a juice and painted all the doors white. Then you chose two quite large prime numbers (but both below 100), and followed the above procedure replacing $2,3$ with the two new prime numbers. When your friend returned, he again found the procedure without much hassle.
So, you got to thinking,
Exactly how did he find out? Is there still a pattern for any of the colours, maybe say purple? If yes, what is it? Changing the starting number of your sequence from 1 to some other number, can you make the door colours patternless? What if instead of 2 primes, you follow this procedure for 3 or more primes, can you get a patternless sequence of colours? What if you replace the prime numbers chosen above, with composite numbers?

Problem 2

Your school is hosting an event and a dinner party afterwards. Knowing that you are good at mathematics, the administration asked you to organise the seating arrangement at the dinner. But you have to follow two rules,
1) You can not fix from beforehand which person sits at which table. Everyone attending the dinner should be able to choose which table they want to join, as long as their preferred table is not completely occupied.
2) you have to make sure that at each table either there is at least 3 people who knows each other from before, or there is at least 3 people who does not know each other. Apparently, dinner conversations are smooth if we follow this rule.
So you got to thinking, what is the least number of people each table should accommodate so that the above rules are followed in each table.

Problem 3

1. Suppose you are given a 4×4 square grid with squares at two diagonally opposite corners removed. In how many ways can you tile this truncated grid with combinations of 2×1 and 1×2 dominoes?



(a) The Truncated 4×4 grid. (b) The dominoes allowed.

Figure 1

2. A knight is in a battle with a mythical dragon with exactly 100 heads. With one swoop of the sword, the knight can cut off either 15, 17, 5 or 20 heads. The dragon dies if all of its heads are cut. But otherwise, in each of the four respective cases, the dragon grows back 24, 2, 14 or 17 heads.

   Can the knight ever defeat the dragon?

3. In a dinner party, $2n$ guests are supposed to sit at a round table, where each seat is next to two others. Each guest has at most $n-1$ enemies among the invited. Is there a seating arrangement where none of the neighbours of each guest is an enemy?