ORDER THEORY

MATHS CIRCLE INDIA–MODULE 20

Countability

A *list* is a collection of elements $a_0, a_1, a_2, \cdots$ that may or may not end. Natural numbers are usually listed as $0, 1, 2, \cdots$. Similarly, we can list integers as $0, 1, -1, 2, -2, \cdots$.

A1 Suppose you are given a list $L_1, L_2, \cdots$ of lists. Show that there is a single list that contains all the elements of all lists.
A2 Show that there is a list of pairs (a, b) of natural numbers.
A3 Is it possible to list all rational numbers?
HW Is it possible to list all real numbers?

Linear Orderings

Suppose there is a set L of candidates and you rank all those candidates according to the following rules:

(1) If x is ranked lower than y and y is ranked lower than z, then x is ranked lower than z;
(2) If x and y are distinct candidates then either x is ranked lower than y or y is ranked lower than x, but not both;
(3) No element is ranked lower than itself.

We say that a set L equipped with such a ranking is a *linear ordering*. The name "linear" is justified because the ranking could be used to imagine candidates as points of a line together with their relative positioning. If x is ranked lower than y, then we write $x < y$ and position x to the left side of y.

B1 Verify that the standard ordering on each of the following sets is a linear ordering:
  (a) the set of natural numbers;
  (b) the set of integers;
  (c) the set of rational numbers.
B2 Can you think of infinitely many rankings on the set of natural numbers?

Consider the following rankings on the set of natural numbers:
$L_1 : 1 < 2 < 3 < \cdots < 0$
$L_2 : 0 < 2 < 3 < \cdots < 1$
$L_3 : \cdots < 2 < 1 < 0$
Then the "shape" of the linear orderings L_1 and L_2 is the same but is different from L_3. In fact, one can get L_2 from L_1 just by swapping 0 with 1. Two linear orderings like this are called *isomorphic*. Any relabelling of elements produces an isomorphic copy.

B3 Can you find infinitely many rankings on the set of natural numbers such that no two of them are isomorphic to each other?
B4 Can you find a ranking of natural numbers that is isomorphic to the standard ordering of integers?
B5 Can you find a ranking of natural numbers that is isomorphic to the standard ordering of rationals?

The standard ordering of natural numbers has a smallest element, namely 0. However, the standard orderings of integers or rationals do not have smallest or largest elements. Also notice that there is a "gap" between two consecutive natural numbers or integers in their standard orderings.

B5' Show that the standard ordering of rationals does not have any gaps.
B6 Suppose you can list all elements of a set L, and have chosen and fixed a ranking of elements of L. Show that for each element x of L, there is a rational $\bar{x}$ such that whenever x is ranked lower than y in L then $\bar{x} < \bar{y}$ in rationals.
B7 A famous result due to Cantor states that if L_1 and L_2 are any rankings of natural numbers without smallest element, largest element or gaps, then they are isomorphic.
  Prove Cantor's result using a modification of the technique used for B6!

Constructing new linear orders from old ones

A countable linear order L is a countable set L equipped with a fixed ranking. Let $\mathbf{n}$ denote any linear order with n elements. The shapes of standard orderings on natural numbers, integers and rationals are denoted ω (omega), ζ (zeta) and η (eta) respectively. For example, if a linear order L has a least element, each element has an immediate successor, and if each element is finitely many steps away from the least element, then its shape is ω.

Suppose L is a countable linear order. The reverse ranking on L is denoted L^*, which we call its *order reversal*.

[C1] Show that ζ^* is isomorphic to ζ.

[C2] Show that η^* is isomorphic to η.

Suppose L_1, L_2 are countable linear orders. If a is an element of L_1, then we write it as $a(1)$ to emphasize that it comes from L_1. Similarly, if b is an element of L_2, then we write it as $b(2)$. Let L denote the countable set containing all such $a(1)$ and $b(2)$. Define a ranking on L where all $a(1)$ are ranked lower than all $b(2)$ in addition to their previous rankings in L_1 and L_2. Such a ranking on L is called the sum of L_1 and L_2, written $L_1 + L_2$. Note that this sum generalizes the usual sum of natural numbers. For example, the sum of $\mathbf{2} : 0 < 1$ and $\mathbf{3} : 0 < 1 < 2$ is $0(1) < 1(1) < 0(2) < 1(2) < 2(2)$, which is isomorphic to $\mathbf{5}$.

[C3] Show that $\mathbf{1} + \omega$ is isomorphic to ω. Is $\omega + \mathbf{1}$ isomorphic to ω?

[C4] Show that $\omega^* + \omega$ is isomorphic to ζ. Is $\omega + \omega^*$ isomorphic to ζ?

[C5] Can you simplify the following expressions?

(1) $\eta + \eta$,
(2) $\eta + \mathbf{1} + \eta$.

[C6] Find a countable linear order L_3, not isomorphic to ω, that has a least element and in which every element has an immediate successor.

Given a countable linear order L, consider the following abbreviations:
$L \cdot \omega = L + L + L + \cdots$;
$L \cdot \omega^* = \cdots + L + L + L$;
$L \cdot \zeta = \cdots + L + L + \cdots$.

[C7] Show that $\mathbf{2} \cdot \omega$ is isomorphic to ω.

[C8] Can you simplify the following expressions?

(1) $\eta \cdot \omega$,
(2) $\eta \cdot \omega^*$,
(3) $\eta \cdot \zeta$.

[C9] Are $\omega \cdot \omega^*$ and $\omega^* \cdot \omega$ isomorphic?

Given elements x, y of a countable linear order L, say that $x \sim y$ if there are only finitely many elements between (endpoints included) x and y in the ranking of L.

[C10] Show the following for elements x, y, z of L:

- $x \sim x$;
- if $x \sim y$ and $y \sim z$, then $x \sim z$;
- if $x \sim y$ then $y \sim x$.

Suppose $(L, <)$ is a linear order and $x \in L$. Let $[x]$, read the *class of* x, denote the set of all elements y such that $x \sim y$. The class $[x]$ contains all elements that are "reachable from" x in a finite number of steps! Let $L^{(1)}$ denote the set of all such classes. Given $x, y \in L$ with $x \not\sim y$, say that $[x] <_1 [y]$ if $x < y$. Indeed, $<_1$ defines a linear ordering on $L^{(1)}$, that is the "blurry vision" version of $(L, <)$!

[C11] Show that $\omega^{(1)}, \omega^{*(1)}, \zeta^{(1)}$ are all isomorphic to $\mathbf{1}$.

[C12] Show that $(\omega + \mathbf{1})^{(1)}$ are $(\omega + \omega^*)^{(1)}$ are isomorphic to $\mathbf{2}$.

[C13] Can you simplify the following expressions?

(1) $(\omega^* \cdot \omega)^{(1)}$;
(2) $(\omega \cdot \omega^*)^{(1)}$;
(3) $\eta^{(1)}$.

Now, can you give a better argument for C9?

[C14] Suppose L is a countable linear order and x is an element of L. Show that $[x]$ is either finite or isomorphic to one of ω, ω^* or ζ.

[C15] Construct all linear orders L satisfying the following properties:

- $L^{(1)}$ has exactly three elements;
- the three classes in $L^{(1)}$ are all different, and are isomorphic to $\mathbf{2}$, ω and ω^*.

[C16] Show that $(\omega + \omega^*) \cdot \omega$ is isomorphic to $\omega + \zeta \cdot \omega$. In other words, $(\omega + \omega^*) + (\omega + \omega^*) + (\omega + \omega^*) + \cdots$ is isomorphic to $\omega + (\omega^* + \omega) + (\omega^* + \omega) + \cdots$

The above exercises indicate that arithmetic of linear orders enjoys several properties of the arithmetic of natural numbers, but it also lacks some! Let us introduce ordered sums and binary products to see what properties hold or fail.

Given a countable linear order L, and a countable linear order L_x for each element x of L, define the *order sum* $\sum_{x\in L} L_x$ as the linear order on elements of the form $y(x)$, where $x \in L$ and $y \in L_x$, defined using "reverse dictionary order" as $y_1(x_1) < y_2(x_2)$ if one of the following happens:

- $x_1 < x_2$ in L;
- $x_1 = x_2$ in L but $y_1 < y_2$ in L_{x_1}.

Note that $L{\cdot}\omega = \sum_{x\in\omega} L$, $L{\cdot}\omega^* = \sum_{x\in\omega^*} L$, and $L{\cdot}\zeta = \sum_{x\in\zeta} L$. Thus, we can define $L_1{\cdot}L_2 := \sum_{x\in L_2} L_1$.

[C17] Suppose L_1, L_2, L_3 are countable linear orders.

(1) Show that $(L_1 + L_2) + L_3$ is isomorphic to $L_1 + (L_2 + L_3)$.
(2) Is $L_1 \cdot L_2$ always isomorphic to $L_2 \cdot L_1$?
(3) Show that $(L_1 \cdot L_2) \cdot L_3$ is isomorphic to $L_1 \cdot (L_2 \cdot L_3)$.
(4) Show that $L_1 \cdot (L_2 + L_3)$ is isomorphic to $L_1 \cdot L_2 + L_1 \cdot L_3$.
(5) Is $(L_2 + L_3) \cdot L_1$ always isomorphic to $L_2 \cdot L_1 + L_3 \cdot L_1$?

[C18] Suppose L is a countable linear order with at least one element. Show that $\eta \cdot L = \eta$.

[C19] In this problem we will show that there are at least as many countable linear orders as many subsets of natural numbers.

(1) Show that $\zeta + \mathbf{1} + \zeta \cdot \omega$ is not isomorphic to $\zeta + \zeta + \mathbf{1} + \zeta \cdot \omega$.
(2) Suppose A is a subset of the set of natural numbers, i.e., A contains some natural numbers. Define $L(A) := \sum_{x\in\omega} L_x$, where $L_x := \begin{cases} \zeta + \mathbf{1} & \text{if } x \text{ is an element of } A; \\ \zeta & \text{if } x \text{ is not an element of } A. \end{cases}$ Show that if A, B are different subsets of the set of natural numbers, then show that $L(A)$ is not isomorphic to $L(B)$.

[HW] Show that a linear order L is isomorphic to $\sum_{[x]\in L^{(1)}} [x]$.

Failure of the pigeonhole principle

The classic pigeonhole principle (PHP) states that if n pigeons are placed into at most $n-1$ pigeonholes then at least one pigeonhole contains at least 2 pigeons. The PHP is formally stated as the following two equivalent statements for a finite set X:

- any onto function from X to X is one-to-one;
- any one-to-one function from X to X is onto.

[HW] Show that the above two statements are equivalent.

We aim to prove a theorem by Dushnik and Miller that states the failure of PHP for linear orders. Here is the statement of the theorem: given a countably infinite linear order L, there is an order-preserving function from L to L that is not onto.

[D1] Prove the theorem if there is an element x of L such that $[x]$ is isomorphic to ω.

Can the same argument be used if $[x]$ is isomorphic to ω^* or ζ for some x?

[D2] Assume that for each x in L, the class $[x]$ in $L^{(1)}$ is finite. Show that $L^{(1)}$ is infinite and has no gaps. Also show that there is a one-to-one order-preserving function from $L^{(1)}$ to L.

[D3] Show that any countably infinite linear order without gaps contains an isomorphic copy of η. Denote such an isomorphic copy of η in $L^{(1)}$ by L_1.

[D4] Show that if y is an element of L_1 and L_2 is obtained by removing y from L_1, then L_2 is also isomorphic to η.

[D5] Use B6 to complete the proof of Dushnik-Miller theorem.

Scattered linear orders

Say that a linear order is *scattered* if it does not contain an isomorphic copy of η.

[E1] Suppose L is a scattered linear order and, for each x in L, L_x is a scattered linear order. Then show that $\sum_{x\in L} L_x$ is also scattered.

Ehrenfeucht-Fraïssé games

Set-up: Positive integer n (number of steps), Linear orders L_1 and L_2.
Players: Player 1 (Spoiler) and Player 2 (Duplicator).
Gameplay: In Step $1 \leq i \leq n$, exactly one of the following happens:

- Player 1 chooses an element a_i of L_1, and then Player 2 chooses an element b_i of L_2.
- Player 1 chooses an element b_i of L_2, and then Player 2 chooses an element a_i of L_1.

Winning Criteria: Player 2 wins the game if for all $1 \leq i \leq j \leq n$, we have $a_i \leq a_j$ in L_1 if and only if $b_i \leq b_j$ in L_2; otherwise Player 1 wins.

[F1] Play 2 rounds of the following games in pairs. Record the outcome of each round in a table.

(1) $n = 2$, $L_1 := \mathbf{2}$, $L_2 := \mathbf{3}$;
(2) $n = 2$, $L_1 := \mathbf{3}$, $L_2 := \mathbf{4}$;
(3) $n = 2$, $L_1 := \omega$, $L_2 := \zeta$;
(4) $n = 3$, $L_1 := \mathbf{7}$, $L_2 := \mathbf{8}$;
(5) $n = 3$, $L_1 := \zeta$, $L_2 := \eta$;
(6) $n = 5$, $L_1 := \omega$, $L_2 := \omega + \zeta$.

We say that Player 2 has a *winning strategy*, written Player 2 $\models G_n(L_1, L_2)$, if every n-step game on orders (L_1, L_2) could be won by Player 2. If Player 2 does not have a winning strategy, then we say that Player 1 has a *winning strategy*, and we write Player 1 $\models G_n(L_1, L_2)$.

[F2] For each game in F1, determine if Player 1 or Player 2 has a winning strategy.

[F3] Show the following for any linear orders L_1, L_2, L_3 and L_4, and any positive integer n:

(1) Player 2 $\models G_n(L_1, L_1)$;
(2) If Player 2 $\models G_n(L_1, L_2)$, then Player 2 $\models G_n(L_2, L_1)$;
(3) If Player 2 $\models G_n(L_1, L_2)$ and Player 2 $\models G_n(L_2, L_3)$, then Player 2 $\models G_n(L_1, L_3)$;
(4) If Player 2 $\models G_n(L_1, L_2)$ and Player 2 $\models G_n(L_3, L_4)$, then Player 2 $\models G_n(L_1 + L_3, L_2 + L_4)$.

[F4] Find small distinct positive integers k, m such that

(1) Player 1 $\models G_2(\mathbf{k}, \mathbf{m})$ and Player 1 $\models G_3(\mathbf{k}, \mathbf{m})$;
(2) Player 2 $\models G_2(\mathbf{k}, \mathbf{m})$ and Player 2 $\models G_3(\mathbf{k}, \mathbf{m})$;
(3) Player 2 $\models G_2(\mathbf{k}, \mathbf{m})$ but Player 1 $\models G_3(\mathbf{k}, \mathbf{m})$;
(4) Player 1 $\models G_2(\mathbf{k}, \mathbf{m})$ but Player 2 $\models G_3(\mathbf{k}, \mathbf{m})$.

[F5] Given positive n, find smallest N such that whenever $k, m \geq N$ then we have Player 2 $\models G_n(\mathbf{k}, \mathbf{m})$.

[F6] Show that for every n, we have Player 2 $\models G_n(\omega, \omega + \zeta)$.

[F7] Suppose L_1, L_2 are linear orders without maximum or minimum elements, and without gaps. Show that for every n, we have Player 2 $\models G_n(L_1, L_2)$.