ORDER THEORY

MATHS CIRCLE INDIA–MODULE 20

Countability

A *list* is a collection of elements $a_0, a_1, a_2, \cdots$ that may or may not end. Natural numbers are usually listed as $0, 1, 2, \cdots$. Similarly, we can list integers as $0, 1, -1, 2, -2, \cdots$.

A1 Suppose you are given a list $L_1, L_2, \cdots$ of lists. Show that there is a single list that contains all the elements of all lists.
A2 Show that there is a list of pairs (a, b) of natural numbers.
A3 Is it possible to list all rational numbers?
HW Is it possible to list all real numbers?

Linear Orderings

Suppose there is a set L of candidates and you rank all those candidates according to the following rules:

(1) If x is ranked lower than y and y is ranked lower than z, then x is ranked lower than z;
(2) If x and y are distinct candidates then either x is ranked lower than y or y is ranked lower than x, but not both;
(3) No element is ranked lower than itself.

We say that a set L equipped with such a ranking is a *linear ordering*. The name "linear" is justified because the ranking could be used to imagine candidates as points of a line together with their relative positioning. If x is ranked lower than y, then we write $x < y$ and position x to the left side of y.

B1 Verify that the standard ordering on each of the following sets is a linear ordering:
  (a) the set of natural numbers;
  (b) the set of integers;
  (c) the set of rational numbers.
B2 Can you think of infinitely many rankings on the set of natural numbers?

Consider the following rankings on the set of natural numbers:
$L_1 : 1 < 2 < 3 < \cdots < 0$
$L_2 : 0 < 2 < 3 < \cdots < 1$
$L_3 : \cdots < 2 < 1 < 0$
Then the "shape" of the linear orderings L_1 and L_2 is the same but is different from L_3. In fact, one can get L_2 from L_1 just by swapping 0 with 1. Two linear orderings like this are called *isomorphic*. Any relabelling of elements produces an isomorphic copy.

B3 Can you find infinitely many rankings on the set of natural numbers such that no two of them are isomorphic to each other?
B4 Can you find a ranking of natural numbers that is isomorphic to the standard ordering of integers?
B5 Can you find a ranking of natural numbers that is isomorphic to the standard ordering of rationals?

The standard ordering of natural numbers has a smallest element, namely 0. However, the standard orderings of integers or rationals do not have smallest or largest elements. Also notice that there is a "gap" between two consecutive natural numbers or integers in their standard orderings.

B5' Show that the standard ordering of rationals does not have any gaps.
B6 Suppose you can list all elements of a set L, and have chosen and fixed a ranking of elements of L. Show that for each element x of L, there is a rational $\bar{x}$ such that whenever x is ranked lower than y in L then $\bar{x} < \bar{y}$ in rationals.
B7 A famous result due to Cantor states that if L_1 and L_2 are any rankings of natural numbers without smallest element, largest element or gaps, then they are isomorphic.
  Prove Cantor's result using a modification of the technique used for B6!

Constructing new linear orders from old ones

A countable linear order L is a countable set L equipped with a fixed ranking. Let $\mathbf{n}$ denote any linear order with n elements. The shapes of standard orderings on natural numbers, integers and rationals are denoted ω (omega), ζ (zeta) and η (eta) respectively. For example, if a linear order L has a least element, each element has an immediate successor, and if each element is finitely many steps away from the least element, then its shape is ω.

Suppose L is a countable linear order. The reverse ranking on L is denoted L^*, which we call its *order reversal*.

[C1] Show that ζ^* is isomorphic to ζ.

[C2] Show that η^* is isomorphic to η.

Suppose L_1, L_2 are countable linear orders. If a is an element of L_1, then we write it as $a(1)$ to emphasize that it comes from L_1. Similarly, if b is an element of L_2, then we write it as $b(2)$. Let L denote the countable set containing all such $a(1)$ and $b(2)$. Define a ranking on L where all $a(1)$ are ranked lower than all $b(2)$ in addition to their previous rankings in L_1 and L_2. Such a ranking on L is called the sum of L_1 and L_2, written $L_1 + L_2$. Note that this sum generalizes the usual sum of natural numbers. For example, the sum of $\mathbf{2} : 0 < 1$ and $\mathbf{3} : 0 < 1 < 2$ is $0(1) < 1(1) < 0(2) < 1(2) < 2(2)$, which is isomorphic to $\mathbf{5}$.

[C3] Show that $\mathbf{1} + \omega$ is isomorphic to ω. Is $\omega + \mathbf{1}$ isomorphic to ω?

[C4] Show that $\omega^* + \omega$ is isomorphic to ζ. Is $\omega + \omega^*$ isomorphic to ζ?

[C5] Can you simplify the following expressions?

(1) $\eta + \eta$,
(2) $\eta + \mathbf{1} + \eta$.

[C6] Find a countable linear order L_3, not isomorphic to ω, that has a least element and in which every element has an immediate successor.

Given a countable linear order L, consider the following abbreviations:
$L \cdot \omega = L + L + L + \cdots$;
$L \cdot \omega^* = \cdots + L + L + L$;
$L \cdot \zeta = \cdots + L + L + \cdots$.

[C7] Show that $\mathbf{2} \cdot \omega$ is isomorphic to ω.

[C8] Can you simplify the following expressions?

(1) $\eta \cdot \omega$,
(2) $\eta \cdot \omega^*$,
(3) $\eta \cdot \zeta$.

[C9] Are $\omega \cdot \omega^*$ and $\omega^* \cdot \omega$ isomorphic?

Given elements x, y of a countable linear order L, say that $x \sim y$ if there are only finitely many elements between (endpoints included) x and y in the ranking of L.

[C10] Show the following for elements x, y, z of L:

- $x \sim x$;
- if $x \sim y$ and $y \sim z$, then $x \sim z$;
- if $x \sim y$ then $y \sim x$.

Suppose $(L, <)$ is a linear order and $x \in L$. Let $[x]$, read the *class of* x, denote the set of all elements y such that $x \sim y$. The class $[x]$ contains all elements that are "reachable from" x in a finite number of steps! Let $L^{(1)}$ denote the set of all such classes. Given $x, y \in L$ with $x \not\sim y$, say that $[x] <_1 [y]$ if $x < y$. Indeed, $<_1$ defines a linear ordering on $L^{(1)}$, that is the "blurry vision" version of $(L, <)$!

[C11] Show that $\omega^{(1)}, \omega^{*(1)}, \zeta^{(1)}$ are all isomorphic to $\mathbf{1}$.

[C12] Show that $(\omega + 1)^{(1)}$ are $(\omega + \omega^*)^{(1)}$ are isomorphic to $\mathbf{2}$.

[C13] Can you simplify the following expressions?

(1) $(\omega^* \cdot \omega)^{(1)}$;
(2) $(\omega \cdot \omega^*)^{(1)}$;
(3) $\eta^{(1)}$.

Now, can you give a better argument for C9?