Math Circle Explorations: Session 2
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Problem 1. Consider a stick of a fixed length L.

(i) A person breaks the stick into some pieces. How do you arrange those pieces so that area enclosed by the arrangement is maximum?

(ii) If you are given the option of breaking the stick into 'n' number of pieces how would you do that so that the area enclosed by them is largest among all possible ways.

(iii) Once you have found the optimal way of breaking the stick into a fixed number of pieces to maximize the area. We can keep increasing the number of pieces and arrange them to get the largest area. How can you compare the largest area obtained with the length of pieces of the stick? Will this process ever end?

(iv) Suppose you are given a rope of length L. How will you join the two ends of the rope so that the area enclosed in the loop is largest? Can we always find a solution to this problem?

(v) What is the largest volume enclosed by a bubble of surface area A?

(vi) Draw a region in a plane such that a unit needle can be placed in it. Can you rotate the needle continuously full 180 degrees within the same region such that it returns to its original position with ends reversed ? Give examples of regions where it is possible to do it.

Problem 2. The students in a classroom have formed numerous clubs. Every club needs to have a president. A student can be a member of multiple clubs. The teacher wants to appoint a president for each club in such a way that no student is the president of more than one club. Clearly, it is not always possible to pick a president for each club while satisfying this requirement. For example, if there are 60 students in class but 61 clubs in all, we cannot pick distinct presidents for each club. So, whether this problem is solvable or not depends on the number of these clubs and also their membership.

How can the teacher figure out whether the problem (of picking distinct presidents for each club) is solvable or not? If it is solvable, can we formulate a systematic method (an "algorithm") by which this problem can be solved?

Problem 3. Given a finite set A of points in space, we denote by $\mathcal{D}(A)$ the set of distances between the points of A. In other words, the members of $\mathcal{D}(A)$ are the numbers $d(x, y)$ where x and y are arbitrary elements of A. Suppose that A has n elements where n is a positive integer.

(a) If all the points of A are on a line, check that $\mathcal{D}(A)$ must have *at least* $n-1$ distinct elements (regardless of how the points are placed). (Check that you cannot guarantee more than $n-1$ elements.)

(b) Now, suppose the elements of A lie in a plane. Can you find a number M (depending on n) such that $\mathcal{D}(A)$ has at least M elements?

(c) Now, suppose that the points of A are no longer in a plane, but in 3-dimensional space. Can you find a number M (depending on n) such that $\mathcal{D}(A)$ has at least M elements?

Problem 4. What is the minimum number of numbers one should select from $\{1, 2, 3, \ldots, 60\}$ to guarantee that there is at least one pair from that selection so that the smaller number of the pair divides the larger number of pair.

Problem 5. Vacillating Mathematician: A mathematician starts walking from her home to the university. She goes a fraction α of the distance and then decides to turn back and walk towards her home. After walking β fraction of the distance, she turns and starts walking towards the university. Again, after α fraction of the distance, she starts returning home and so on. Where will she reach (if at all) eventually?

Problem 6. Consider the vacillating mathematician problem above. Now suppose after walking α fraction of the distance from her home to the university, she tosses a biased coin with probability of heads $= p$. If she gets a heads, she continues to walk towards the university but if she gets a tail she decides to walk back towards her home. After walking α fraction of the distance between her current position and the destination, she again flips a coin and either continues in the same direction or goes in the opposite direction, and so on.