Congruent Numbers

A right triangle is called rational when its legs and hypotenuse are all rational numbers (Examples of rational right triangles include Pythagorean triples like (3, 4, 5)).

Definition(**Congruent number**): A positive rational number n is called a *congruent number* if there is a rational right triangle with area n: there are rational $a, b, c > 0$ such that $a^2 + b^2 = c^2$ and $(1/2)ab = n$.

The congruent number problem makes its earliest appearance in an Arab manuscript traced to the tenth century and around 1000 years old.

1 Recall of Pythagoreal triples

Definition 1.1 *Let X, Y, and Z be rational numbers. We say (X, Y, Z) is a Pythagorean triple if $X^2 + Y^2 = Z^2$. If $X, Y, Z \in \mathbb{Z}$ and gcd$(X, Y, Z) = 1$ we say (X, Y, Z) is a primitive Pythagorean triple.*

Remark 1.2 *Why!*
Let (X, Y, Z) be a primitive Pythagorean triple. Then there exists $m, n \in \mathbb{N}$ so that $X = 2mn$,, $Y = m^2 - n^2$ and $Z = m^2 + n^2$. Conversely, any $m, n \in \mathbb{N}$ with $m > n$ define a right triangle.

This remark allows us to construct as many congruent numbers as we want. Namely, for any $m, n \in \mathbb{N}$ we have that $N = \frac{1}{2}(2mn)(m^2 - n^2)$ is a congruent number. Following table is an example:

. Congruent numbers from Pythagorean triples

m	n	X	Y	Z	N
2	1	4	3	5	6
3	1	6	8	10	24
3	2	12	5	13	30
4	1	8	15	17	60
4	3	24	7	25	84
4	2	16	12	20	96
5	1	10	24	26	120
5	4	40	9	41	180

We would like to deal with the cases of rational right triangles also. Given a right triangle with integer sides X, Y, and Z and congruent number $N = a^2 N_0$, we can form a right triangle with rational sides and congruent number N_0 by merely dividing X and Y by a.

Remark 1.3 *Why!*
Enough to study positive integers that are square-free.

Using this simple technique, one can find the following table with rational sides:

Congruent numbers from rational right triangles

X	Y	Z	N
3/2	20/3	41/6	5
4/9	7/4	65/36	14
4	15/2	17/2	15
7/2	12	25/2	21
4	17/36	145/36	34

Using the method described thus far, if we cannot find a triangle with area N, it does not mean N is not congruent. It may just be that we have not looked hard enough to find the triangle. For example, the integer 157 is a congruent number with sides $X = \frac{6803298487826435051217540}{411340519227716149383203}$ and $Y = \frac{411340519227716149383203}{21666555693714761309610}$.

2 Questions

Question 2.1 *The number 1 is not congruent.*

Question 2.2 *A number n is congruent if and only if there exists a rational number a such that $a^2 + n$ and $a^2 - n$ are both squares of rational numbers.*

Question 2.3 *(BONUS)*
i) Using 1 is not congruent, prove that $\sqrt{2}$ is irrational.
ii) If there are non-zero integers X, Y, Z such that $X^4 - Y^4 = Z^2$, then 1 is a congruent number. In other words, Fermat's last theorem is true for $n = 4$ (using previous questions).