Math Circle Explorations
IISER Mohali
November 3, 2023

---

**Problem 1.** A continued fraction is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

where $a_0$ is an integer, and all the other $a_i$s are natural numbers. The dots indicate the presence of many more $a_n$s, possibly infinitely many, that are not being written. We denote this continued fraction as $[a_0; a_1, a_2, a_3, \cdots]$.

We illustrate with two examples.

$$3.15 = 3 + \cfrac{1}{6 + \cfrac{1}{1 + \cfrac{1}{2}}}, \qquad -\frac{3}{7} = -1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3}}}.$$

That is, as per our notation, $3.15 = [3; 6, 1, 2]$ and $-\frac{3}{7} = [-1; 1, 1, 3]$. The continued fraction for $\pi$ looks like $[3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, \cdots]$.

(a) Can you think of a way of converting a real number into a continued fraction?

(b) Observe that our two initial examples both produce finite continued fractions. Which real numbers do you think will have this same property? Can you find an explanation for this?

(c) Find the continued function expressions for the following real numbers

- (i) $1 + \sqrt{2}$
- (ii) $\sqrt{2}$
- (iii) $\sqrt{5}$
- (iv) $\frac{1 + \sqrt{5}}{2}$

(d) For which real numbers, will the continued fraction representation eventually form a repeating pattern? (For example, a sequence like $[2; 3, 2, 3, 2, 3, \cdots]$ or $[-4; 5, 7, 2, 3, 2, 3, 2, 3, \cdots]$.)

(e) For a continued fraction $[a_0; a_1, a_2, a_3, \cdots]$, let us denote the rational number corresponding to the truncation $[a_0; a_1, a_2, a_3, \cdots, a_k]$ as $\frac{p_k}{q_k}$ [Without reducing the fraction]. Can you think of a recurrence relation for $p_n$ and $q_n$?

(f) Are $p_k$ and $q_k$ relatively prime for all $k$?
**Hint:** Try to find a relation between $p_k$, $q_k$, $p_{k-1}$ and $q_{k-1}$.

**Problem 2.** Let $n$ be a positive integer. We would like to find a *finite* collection $S$ of points in the plane such that every point of $S$ is at unit distance from exactly $n$ points of $S$. For example, if $n = 2$, the vertices of an equilateral triangle with each side of unit length form such a set. For which positive integers $n$ can we construct such a set?

**Problem 3.** Let $D$ denote the closed unit disc in the plane. In other words, $D$ is the set of all points that are at distance $\leqslant 1$ from the origin. Is it possible to partition $D$ into two subsets $R$ and $B$ which are congruent?

[*Notes:*

(i) Two subsets of the plane are *congruent* if one can be made to lie on the other exactly after applying a series of distance preserving operations like rotations, reflections and translations.

(ii) The sets $R$ and $B$ *partition* $D$, i.e. $R \cup B = D$ and $R \cap B$ is empty. You may think of $R$ as the set of points that are being coloured red and $B$ as the set of points that are being coloured blue. All points of $D$ are being coloured and each point gets exactly one colour. ]