Math Circle Explorations
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October 20, 2023

Problem 1. Let n be a positive integer. Suppose we are given a set $\mathcal{S}$ of n points in the plane. We assume that these points are not collinear. Let us consider the set $\mathcal{L}$ of all the lines that contain two or more points of $\mathcal{S}$. We will investigate various questions regarding this set of lines.

(a) A line in $\mathcal{L}$ is said to be *ordinary* if it contains only two of the points of $\mathcal{S}$. Prove that there exists at least one ordinary line.

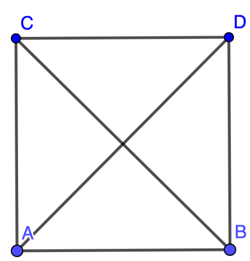
(b) Obtain a lower bound for the number of lines in $\mathcal{L}$.

(c) Obtain a lower bound for the number of ordinary lines in $\mathcal{L}$.

(d) Two lines have the same *slopes* if they are parallel. How many different slopes do the lines of $\mathcal{L}$ determine? Clearly the answer depends on our choice of $\mathcal{S}$. Can we find a lower bound on the number of slopes for various choices of $\mathcal{S}$. (Note that $\mathcal{S}$ is subject to the restriction that the points of $\mathcal{S}$ are not all collinear.)

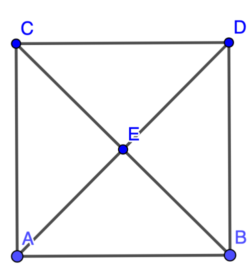
For example, it is easy to see that if $n = 3$, then for any choice of $\mathcal{S}$, there are at least 3 slopes.

If $n = 4$, there exists a configuration with four slopes. For example, in the following configuration we we have the following grouping of lines according to slope: $\{AB, CD\}$, $\{AC, BD\}$, $\{AD\}$ and $\{BC\}$.



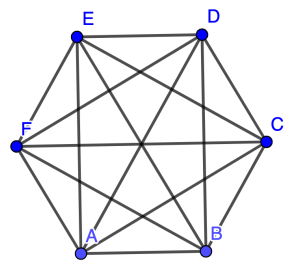
Can you show that you will at least find 4 slopes if $n = 4$?

If $n = 5$, there exists a configuration with four slopes. For example, in the following configuration we we have the following grouping of lines according to slope: $\{AB, CD\}$, $\{AC, BD\}$, $\{AD\}$ and $\{BC\}$.

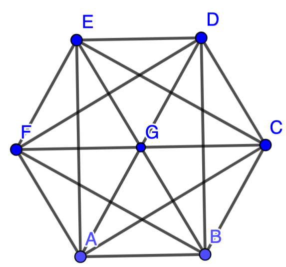


Again, can you show that you will find at least 4 slopes if $n = 5$? (This is actually obvious if you solved the case $n = 4$ above!)

If $n = 6$, consider the configuration of vertices of a regular hexagon. Verify that there are 6 slopes in this configuration.



If $n = 7$, consider the configuration consisting of the six vertices of a regular hexagon and its center. Verify that this configuration has 6 slopes.



Can you formulate a formula or at least a lower bound for the minimum number of slopes in terms of n?

Problem 2.

(a) Let us begin with a rectangular card (say of credit card size, which is 5×8 in centimeters, approximately) whose one side is black and other white. An identity card is made from the same by punching small holes (say of diameter 1) in a corner touching two sides of the card.[1]

- How many different identity cards can you make if you are allowed to punch any number of such holes (maximum four holes are possible if there were four corners)?
- What if the card was square (say 3×3) to begin with?
- What if we are also allowed to punch a hole at the midpoint of each edge?
- What if we are also allowed a punch a hole right at the centre?
- What if the card was black on both sides to begin with?

(b) Let us try to find some "formulae" instead. Instead of beginning with a card which is square, let us begin with a card which is shaped as a regular n-gon, each edge being of size 3. It is still black on one side, white on another. We can make id cards, as before, by punching a small hole (diameter 1) in the corner which touches two of the sides. If one is allowed to punch an arbitrary number of holes (maximum n such holes are possible), how many such id cards are possible? Give this in terms of n.

(c) What is the corresponding formulae when the card has both sides black?

[1] The exact size is not important in this problem, just the relative orientation is.