Math Circle Explorations
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Problem 1. A king wants to connect 10 of the biggest cities in his kingdom with flights. A flight route connects exactly one pair of cities. For each flight route that he plans, he assigns an aeroplane to fly back and forth along this route. He wants to make sure that every pair of these 10 cities connected by a sequence of flights.

(a) What is the least number of aeroplanes that the king needs to buy?

(b) In how many ways can he plan the flight routes using the least number of planes?

(c) What if the number of cities is equal to n where n is some positive number?

Problem 2.

(a) A map of India is placed flat on the ground in Mumbai. A geographer claims that there exists a point on the map that lies exactly over the point on the ground which it represents on the map. Can you find a mathematical argument to support his claim?

(b) What if we fold the map before we place it on the ground? What if we crumple it up in some random way? Is the claim in part (a) still true?

(c) Let us see if we can notice a pattern from the discussion in (a) and (b). Let $\mathcal{C}$ be the set of all points (i.e. locations) in India. For two locations P and Q, we will denote the distance between P and Q by $dist(P, Q)$.

A map of India lying flat on the ground can be thought of as a function $f : \mathcal{C} \to \mathcal{C}$. For any location P in the country, $f(P)$ is the point of the map representing the location x. The function f has the property that it shrinks the distances between points by a constant factor. In other words, there exists some constant $c < 1$ such that for any two points P and Q of $\mathcal{C}$,

$$dist(f(P), f(Q)) = c \cdot dist(P, Q).$$

Note that c is less than 1, i.e. f shrinks the distance between points! The claim in part (a) states that there exists a point P such that $f(P) = P$. See if you can formulate some variations of this statement and check if they are true.

For example, let D be a set of points in the $x - y$ plane which are at distance less than or equal to 1 from the origin. (Thus, D is a disc of radius 1, including the boundary.) Let $f : D \to D$ is a function such that for any two points P, Q, we have $dist(f(P), f(Q)) = (1/2) \cdot dist(P, Q)$. Show that there exists a point x in D such that $f(x) = x$.

(d) Does the claim in part (c) hold if we replace D by other subsets of the plane? For example, what if f is a function from the entire plane into itself such that

$$dist(f(P), f(Q)) = (1/2) \cdot dist(P, Q).$$

Does there exist a point in the plane such that $f(P) = P$?

What happens if we replace D in part (c) by the *open disc*, i.e. the part of the disc that does not include the boundary? (This is the collection of all points which are at a distance strictly less than 1 from the origin.)

(e) Explore further generalizations of the result in part (c) and see if they are true.

Problem 3.

(a) How many fractional expressions (not necessarily reduced) are there whose denominator is less than 10? (For integers p/q, we say that the fractional expression p/q is *reduced* if p and q are coprime. For example, $4/10$ is not a reduced expression, but $2/5$ is reduced.)

(b) How many rational numbers between 0 and 1 have denominator less than 10?

(c) Start writing all the positive rational numbers between 0 and 1, both included, in the following way. First, write the rationals with denominator 1 in increasing order. Then, insert the rational numbers with denominator 2 in this list in the correct places so that the list remains increasing. Then do the same thing for rational numbers with denominator 3, then for rational numbers with denominator 4. At this point, the list should look like

$$\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}.$$

If we keep going like this, which number will appear first between $\frac{1}{2}$ and $\frac{2}{3}$? What about $\frac{2}{3}$ and $\frac{3}{4}$?

(d) Based on the answers above, would you like to predict what number will appear first between $\frac{a}{b}$ and $\frac{c}{d}$, if they are neighbors in the list at some stage?

(e) Try to prove the conjecture you made for part (d).