Math Circle Explorations
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Problem 1. You are familiar with the notion of the decimal representation of any real number. For example, we write $1/10 = 0.1$, $1/3 = 0.333333\ldots$, $\sqrt{2} = 1.41421356\ldots$, etc. The "..." indicates that there are many more digits, sometimes infinitely many, that are not being written.

(a) Can you precisely describe how the decimal representation of a real number can be found? When we write

$$1/3 = 0.33333\ldots,$$

in what sense is the left hand side "equal" to the right hand side?

(b) Observe that the decimal representation of some numbers shows repeating patterns. For example,

$$1/3 = 0.33333\ldots$$

has the digit 3 appearing over and over, infinitely many times. The decimal representation of $1/7$ can be found to be

$$1/7 = 0.142857142857142857\ldots.$$

Thus, the sequence 142857 is repeating infinitely many times. (How can we be sure that it is repeating forever. Can you make sure that it does not suddenly change later?) On the other hand, the decimal representation of $\sqrt{2}$ does not show any patterns. Which numbers do you think will have repeating patterns in their decimal representation? Whatever your guess might be, can you find an explanation for this?

(c) Suppose a number α has a repeating pattern in its decimal representation. How big is the repeating pattern? For example, in the decimal representation of $1/3$, we see a small repeating pattern consisting of the single digit 3, but in the decimal representation of $1/7$, we see the repeating pattern 142857 consisting of 6 digit. Is the length of the repeating pattern related to some property of the number α?

Problem 2. An ant is walking on a plane surface. Its movement is *continuous*, i.e. it does not jump from one point to another. It is walking at a constant speed. It is not necessarily moving in a straight line. The path is *curved*, and the *curvature* of the path varies from point to point. But what do we mean by *curvature*?

(a) How would you measure the curvature of the ant's path? If the ant is moving in a straight line, we would like to say the curvature is 0. If the ant is moving along a circle of unit radius, we would like to say that the curvature along such a path is equal to 1. The curvature of a large circle should be small and the curvature of a small circle should be large. Can you come up with a notion which fits all these requirements? (Hint: Try to first define the notion of curvature for circles and then approximate a given path with circles.)

(b) If the ant is moving at constant speed, its path is curved only if its direction is changing. In other words, its *velocity* is changing. In other words, its *acceleration* should be non-zero. Can you relate the notion of curvature from part (a) to the acceleration?

(c) Suppose that the ant is moving along a polygonal path. In other words, the path consists of straight line segments for the most part, but it suddenly changes its direction at some points. Suppose that the path is closed, i.e. the ant returns to its original position after some time. What can you say about the total change in its direction through its entire trip? (Hint: What is the sum of the external angles of a polygon?)

(d) Suppose that the ant completes a trip along a closed path that is not necessarily polygonal. Can you approximate this path using polygonal paths, and use your observations in part (c) to obtain a result about the curvature of this path?

Problem 3. (a) Three players are playing a game. At the start of the game, each of them is given a coin. In each round, every player gives away one of their coins (or none if they don't have any at the beginning of said round) to one of the other two players at random. What is the probability that each of them has a coin at the end of 100 rounds?

(b) A switch is kept in a room. It is turned on at the start. Then, 100 people go into the room one by one. When a person enters the room, if they find the switch to be on, they flip a coin, and if it comes up head they switch it off. However, if they find it off, they roll a dice, and if it comes up 6, they switch it on. What is the probability that the switch will be on after the 100 people have entered the room?

(c) Three players, Anita, Abdul, and Amit are playing the same game as mentioned in part (a). However, only Anita has a coin at the start of the game, and the other two have none. What is the probability that Anita has the coin at the end of 100 rounds?