Maths Circle India

TIFR-STCS Maths Circle Team

Session 8: August 25, 2023

1 Sports day: A promise

A year has passed and Arun, Barun and Kiron ("ABK") have again been asked to schedule the sports day. The situation is exactly the same as before, but this time they are wondering if they can schedule more games by exploiting the fact that while some games might have a lot of conflicts, other have very few. More precisely, they are wondering what is the best they can do if there are n games, where the ith game (where $1 \leqslant i \leqslant n$) has exactly c_i conflicts. They read in a comment in a book that in this case, they should be able to schedule at least

$$\sum_{i=1}^{n} \frac{1}{c_i + 1} \tag{1}$$

games. But the comment section in the book was too small to either give a proof, or a method for finding such a set of games.

Suppose first that what the book is claiming is actually correct. Can you come up with lists where what the book is promising is much better than what we did last time? In particular, can you find a list where what is promised above is at least twice as good as what we had last time? At least 100 times as good? (Last time, we could show that if $d_i \leqslant D$ for every i, then we can schedule at least $n/(D+1)$ games.)

In the rest of this exploration, we will try to prove what the book claimed. But before that, let us take a detour and try to analyse "randomly" ordered lists.

2 "Randomly" ordered lists

1. Suppose that there are n games in ABK's list. Show that there are a total of

   $$n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

   possible orderings of the list. This number is usually written as $n!$ (and sometimes as $\underline{\lfloor n}$, especially in older books), and called "n factorial".

2. Suppose that one of these games is *ābol tābol*. What is the number of orderings in which *ābol tābol* is ranked first? What is the *fraction* of orderings in which *ābol tābol* is ranked first?

3. Can you derive the value of the fraction above *without* actually counting the total number of orderings like we did above? (Hint: Suppose we fix any *arbitrary* ordering of the other games. What are the possible ranks of *ābol tābol*? Is each of these ranks obtained in a *different* ordering?)

4. Now suppose that we only care about the *relative* rank of *ābol tābol* and *tābol ābol*. That is, we only care about which one of these two ranks ahead of the other, without worrying about the ranks of the other games in the list. Now, what is the fraction of lists in which *ābol tābol* ranks ahead of *tābol ābol*?

5. Now suppose that we only care about the *relative* rank of *ābol tābol*, *tābol ābol* and *ônyô khelā*. That is, we only care about which one of these three games ranks ahead of the other two, without worrying about the ranks of the other games in the list. Now, what is the fraction of lists in which *ābol tābol* ranks ahead of the other two?

6. In general, suppose that we only care about the relative rank of some D games $g_1, g_2, g_3, \ldots, g_D$, irrespective of the ordering of the other $n - D$ games (of course, $D \leqslant n$). What is the fraction of orderings in which g_1 ranks ahead of $g_2, g_3, \ldots, g_D$?

7. Fix a game g_i, which conflicts with c_i other games. What is the fraction of orderings in which g_i ranks above all the games it conflicts with?

A linguistic interlude It is useful to translate the above results into a slightly different language. We assign a *weight* $w(P) = \frac{1}{n!}$ to each ordering P. Note that the weights are non-zero and sum up to 1 (such weights are called "probabilities"). The fraction of orderings in which, for example, g_1 ranks above $g_2, g_3, \ldots, g_D$ is then just the total weight of all orderings in which the "event" that "g_1 ranks above $g_2, g_3, \ldots, g_D$" occurs. This fraction is then also called "the *probability* that g_1 ranks above $g_2, g_3, \ldots, g_D$" (or, with more verbosity, "the *probability* that the *event* that g_1 ranks above $g_2, g_3, \ldots, g_D$ occurs").

3 Peaking games

Now, given an ordering O, we say that the game g is *peaking* in O if it has a higher rank in O than all the games it conflicts with. For an ordering O and game g, we define $\mathrm{peak}(O, g) = 1$ if g is peaking in O and $\mathrm{peak}(O, g) = 0$ otherwise. We denote the total number of games that are peaking in O as $\mathrm{peak}(O)$. Thus, for every ordering O,

$$\mathrm{peak}(O) = \sum_{i=1}^{n} \mathrm{peak}(O, g_i).$$

Let us define AvgPeak to be the average number of peaking games in a ordering, where the average is taken with respect to the above weights. Thus,

$$\mathrm{AvgPeak} = \sum_{O} w(O)\,\mathrm{peak}(O),$$

where the sum is over all possible orderings O.

Similarly, let us define $\mathrm{AvgPeak}(g)$ to be the *fraction* of orderings in which the game g is peaking.

1. Fix any ordering O, and let g and h be distinct games that are both peaking in O. Can g and h have a conflict?

2. Show that for any game g,

$$\mathrm{AvgPeak}(g) = \sum_{O} w(O)\mathrm{peak}\,(O, g)\,,$$

   where the sum is over all orderings O. What is the value of $\mathrm{AvgPeak}(g)$ in terms of the number of games with which g conflicts? (Hint: You already computed this above.)

3. Show therefore that

$$\mathrm{AvgPeak} = \sum_{i=1}^{n} \mathrm{AvgPeak}(g_i).$$

4. Can you argue that there must exist an ordering Q for which

$$\mathrm{peak}\,(Q) \geqslant \mathrm{AvgPeak}?$$

5. Can you now prove the claim that ABK found in the book?

4 Existence is not enough

ABK are, of course, quite happy that they have finally proved the claim in the book.[1] Unfortunately, however, it doesn't help with their job to just *know* that such a set of game *exists*. They actually need to *find* a large set of non-conflicting games and give that list to the school, so that the school knows which games to schedule.

Can you find a reasonably "fast" procedure that will actually find a set of non-conflicting games that is at least as large as what was promised in eq. (1) above?

[1]

Don't just read it; fight it! Ask your own questions, look for your own examples, discover your own proofs.

– Paul Halmos