Maths Circle India

TIFR-STCS Maths Circle Team

Session 7: August 11, 2023

1 Sports day

Arun, Barun and Kiron have been asked to help organize the annual sports day at their school. They have a large list of n games, out of which they have to choose some subset of games to organize. However, for various reasons, there are *conflicts* between the games. For example, the game of *ābol tābol* cannot be included in the program if the game of *tābol ābol* is also included (perhaps because the games are too similar). Arun, Barun and Kiron are given a list of such conflicts: each conflict is given as a pair of games that cannot both be included in the program.

They are wondering what is the maximum number of games they can organize, given their lists of games and conflicts. At this point, Arun and Barun, who have been poring over the lists, hit upon an observation: no game conflicts with more than one other game. Then the three of them sit quiet for a few minutes. Suddenly, Kiron exclaims, "Then, we must be able to organize at least $n/2$ games!"

What does Kiron have in mind? Is $n/2$ necessarily the best possible in every situation of the kind ABK have? Could there be a situation where you cannot do better than what she has in mind?

What if no game conflicted with more than *two* other games? *three* other games? *ten* other games?

2 New dice from old

Nala and Damayantī want to play their favourite ancient version of *Snakes and Ladders*, which requires somewhat specialized dice. Unfortunately for them though, they packed only a standard six-faced die, and they are traveling through a dense forest. So they wonder: could they somehow use their trusty six-faced die to simulate these fancier dice?

Can you simulate the following with a six-faced die, perhaps by throwing it many times?

1. A two-faced die (i.e., a coin)
2. A three-faced die
3. A four-faced die
4. A twelve-faced die
5. An eight-faced die
6. A seven-faced die

3 Some hard work for later

Can you find positive integers m and n, with $m > n + 1$, such that

$$\frac{1}{m} + \frac{1}{n} \leqslant \frac{1}{m-1} + \frac{1}{n+1}?$$

Now, how small can you make the sum

$$\frac{1}{a} + \frac{1}{b},$$

where a and b are positive integers satisfying $a + b = N$ (where N is another positive integer)? To make the game concrete, what do you get for $N = 10$ and $N = 9$?

Now, how small can you make the sum

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c},$$

where a, b and c are positive integers satisfying $a + b + c = N$ (where N is another positive integer)? To make the game concrete, what do you get for $N = 10$ and $N = 9$? What about $N = 100$? $N = 1000$? $N = 10000$?