

Maths Circle India: Module 8, Session 3
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1 Greatest Common Divisor

Suppose a and b are two positive integers. A positive integer d is called the *greatest common divisor* (gcd) (also known as *highest common factor* or hcf) of a and b if

- d divides both a and b ;
- if a positive integer c divides both a and b , then c divides d .

(Here m divides n means n is divisible by m .)

- (i) Assume that $a > b$. We can find integers q_0, r_0 such that $a = q_0b + r_0$, where $q_0 \geq 1$ and $0 \leq r_0 < b$. If $r_0 \neq 0$, we then find integers q_1, r_1 such that $b = q_1r_0 + r_1$, where $q_1 \geq 1$ and $0 \leq r_1 < r_0$. Again if $r_1 \neq 0$ we divide r_0 by r_1 and get remainder r_2 , and so on. This process eventually terminates (**Why?**), and we get $r_{n-2} = q_n r_{n-1} + r_n$, and finally $r_{n-1} = q_{n+1} r_n$.

- Show that r_n divides both a and b .
- If c is a common divisor of a and b , then show that c divides r_n .

In particular, according to the definition of gcd given above, r_n is the gcd of a and b . ***This will prove that the Euclidean algorithm of finding gcd actually works.***

- (ii) Let $a > b$. Prove that the gcd of a and b is the same as the gcd of $a - b$ and b .

2 Lowest Common Multiple

Formulate a definition of the *least common multiple* (lcm) of two positive integers a, b (mimicking the definition of gcd given above). If d is the gcd and l is the lcm of a, b , then show that $ld = ab$.

3 Measurement of Milk

Suppose there is unlimited supply of milk and a huge empty container with unlimited capacity.

If you have two buckets of capacities 7 litres and 3 litres, respectively, then can you measure and give a customer exactly 5 litres of milk?

Find all positive integers m such that you can measure and give exactly m litres of milk.

Think: In general, if you have two buckets of capacities a litres and b litres, respectively (with a and b positive integers), then what is the answer to the second question above?