Math Circle Explorations: Session 3

Room A

Problem 1. Observe that for some small integers, if we attach some extra digits on the right hand side, we can "complete" them to a power of 2. For example, for the integer 1, we write 6 on the right to get $16 = 2^4$. Similarly, we write 56 to the right of 2 to get $256 = 2^8$. Some more examples:

$$3 \rightsquigarrow 32 = 2^5$$
$$4 \rightsquigarrow 4096 = 2^{12}$$
$$5 \rightsquigarrow 512 = 2^9$$
$$6 \rightsquigarrow 64 = 2^6 \ldots$$

Seems easy enough... right? Next one is a bit harder.

$$7 \rightsquigarrow 70{,}368{,}744{,}177{,}664 = 2^{46}$$

That took some work!

$$8 \rightsquigarrow 8192 = 2^{13}$$
$$9 \rightsquigarrow 9007199254740992 = 2^{53}$$

Almost gave up there!

$$10 \rightsquigarrow 1024 = 2^{10}$$
$$11 \rightsquigarrow 1125899906842624 = 2^{50}$$
$$12 \rightsquigarrow 1208925819614629174706176 = 2^{80}$$

Phew!

$$13 \rightsquigarrow 131072 = 2^{17}$$

... and so on.

Do you think that *any* integer can be "completed" to a power of two by writing some more digits to the right? Or is it impossible to do this for some integers?

Problem 2. Let x be a real number greater than 1. We consider the sequence $\lfloor x \rfloor, \lfloor 2x \rfloor, \lfloor 3x \rfloor, \ldots$ and so on. (For any real number y, $\lfloor y \rfloor$ denotes the greatest integer less than or equal to y. Thus, $\lfloor 2.73 \rfloor = 2$, $\lfloor 3 \rfloor = 3$.) Let us denote this sequence by S_x.

How do these sequences behave? Given real numbers x and y, both greater than 1, is there any way to predict if S_x and S_y will have some common terms? Is there any way to predict if they will have no common terms?

Room B

Problem 1. Do the following:

a) Let θ be a rational number between $(0, 360)$. Consider a circle of unit radius centred at origin in the plane. Start with the point $(1, 0)$ and rotate it in the anticlockwise direction by θ degrees. Let us again rotate the new position by θ degrees and repeat this process. Will you ever hit the initial positon $(1, 0)$ if at each step you rotate by θ degrees? Will you hit the initial position multiple number of times?

b) Now pick θ to be any irrational number between $(0, 360)$. Let us proceed in a similar manner as in the previous part. Will you ever hit the initial position $(1, 0)$ in this case?

c) Let S be the collection of all points on rotation by θ degrees starting from $(1, 0)$ as above. Pick any interval I having more than one point on the unit circle centred at the origin. When will be the intersection non-empty for any interval I with S on the given circle? Does the choice of θ matter here?

Problem 2.

1. Can you find a collection of "countable" points on the sphere of radius 1 such that those points lie in any patch you cut out of sphere?

2. Can you find a way of rotating a unit needle continuously by 180 degrees in plane such that the area covered is small? You are allowed to move the point continuously about which you rotate the needle along with the rotation as well.

Room C

Problem 1.

1. Suppose n points are arranged in a circle, in a circular order, each the same distance from the previous one. A frog sitting on one of these points makes a jump of distance d every minute. The direction in which the frog jumps and the distance d are constant. You get to make one guess every minute where the frog is (knowing neither d, the direction, or the initial position), and you win if you can guess correctly. Can you give a strategy so that you eventually win?

2. Instead of n points in a circle, suppose there are infinite points arranged uniformly in order (think about integers on the number line). Can you still device such a strategy?
   What if we have points with integer co-ordinates in the whole plane or the whilespace (our naive frog still jumps same distance every minute, and in the same direction)?

3. What if we begin with the the rational numbers instead of integers on the number line? What if we have all the points on the real line?

Problem 2. We call an infinite set "countable" if it can be counted, that is we can arrange all the elements in an order and say a_1 is the first element, a_2 is the second element, ... and so on (this list should exhaust all elements!). Consider all subsets (that is the power set) of natural numbers $\mathbb{N}$. Is it countable?