Maths Circle Explorations: Session 8

IISER Mohali

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Problem 1

A few questions about finding patterns in colourings:

1. Suppose we colour all the integers with two colours, red and blue. Verify that regardless of how we choose the colouring, we always find three integers which are the same colour and are evenly spaced? (By "evenly spaced", we mean that the integers should be of the form a, $a+d$ and $a+2d$ where d is a positive integer.)

2. We say that the n numbers $a_1, a_2, \ldots, a_n$ are in *arithmetic progression* if they are of the form $a, a+d, \cdots a+(n-1)d$. So, part (1) is about finding arithmetic progressions of length 3, such that all the terms have the same colour, i.e. they are *monochromatic*. Is it possible to find longer arithmetic progressions which are monochromatic? Or is it possible to colour the integers in such. a way that monochromatic arithmetic progressions of a certain length are avoided?

3. Is it possible to find monochromatic progressions of *infinite length*?

4. What happens if we increase the number of colours?

5. The integers can be visualized as points on a line. So, the above questions refer to colouring points one lines. What if we colour points in a plane or in three-dimensional space? What kind of monochromatic patterns can we hope to find (or not find)? For example, can we always find a monochromatic equilateral triangle? What about other patterns?

Problem 2

1. Suppose we have a photograph of a straight running track, call it AB, whose length we want to compute. To do this, we identify some known objects $\{X_i\}$, which we think of as points on the track; and we ask people on the ground to measure some of the distances X_iX_j. How many of X_iX_j should we know apiori to compute the length of the track?
   Note: We do have a scale and can measure distances on the photograph itself.

2. Suppose instead that the points X_i were not on the running track but on the playground (which we assume to be a plane) containing the track, then how many of the distances X_iX_j do we need? What if the playground is not a plane and we instead have golf-course with significant highs and falls (we still assume the track is a straight line)?

3. The playground is used for a race, and you have a video of two runners practicising on two parallel straight tracks. You have no one on the ground to measure any distance. Can you tell which of the runner is faster? (We visualize runners as points on the straight track, and they run with a uniform speed, but they are not assumed to start together).

Problem 3

1. Amit and Smita play a series of games where in each game they can either win or lose one rupee. In each round, Amit earns Rs. 1 with probability $p \in (0,1)$, and in this case Smita loses Rs. 1. Conversely, Amit loses Rs. 1 with probability $q = 1 - p$ and Smita gains Rs. 1. Suppose Amit starts with Rs. $x > 0$ and Smita starts with Rs. $y > 0$. What is the probability that Amit loses all his money *eventually*?

2. A person X walks on $\mathbb{Z}$ starting at 0. At every time-step, she moves one step towards the right with probability p and one step towards the left with probability $q = 1 - p$.

   (a) What is the probability that X will return to 0 in exactly n steps?

   (b) Take $p = 1/2$. Suppose she is at r at time n. What is the probability that she has not returned to 0 so far?