Maths Circle Explorations: Session 4

TIFR, Mumbai

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Problem 1

Suppose we have 1000 doors in a row marked $1, 2, 3, ..., 999, 1000$. All doors are closed. Now, we pick a number n lying between 1 and 1000, and we change the condition of the doors (opening it if it is closed, and closing it if it is open), whose numbers are divisible by n. To elaborate, initially all the doors were closed. We pick 1, and as every number is divisible by 1, we open all the doors. Then we pick 2, and as only even numbers are divisible by 2, we close the even numbered doors, leaving the odd numbered doors open. We continue this process for $3, 4, 5, ..., 1000$.

Now the question is, how many doors are open and how many are closed at the end of this process?

Problem 2

Imagine yourself to be a farmer in prehistoric times when the concept of numbers hadn't been discovered yet. Suppose you have 10 cows (note: the concept of 10 hadn't been discovered yet) and a fellow farmer has 17 cows. How would you naturally come up with a way to tell that the other farmer has more cows than you have?

By the same line of thought, how would you come up with a way to tell that you have as many eyes as you have ears? (note again : the concept of 2 hadn't been discovered yet)

Can you now come up with an answer to the following questions:

1. Are there as many even numbers as odd numbers?
2. Are there as many natural numbers

   $$\mathbb{N} := 1, 2, 3, \ldots$$

   as integers

   $$\mathbb{Z} := \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots?$$

3. Are there as many Integers as rational numbers. Rational numbers are written as p/q where p and q are integers and $q \neq 0$?
4. Are there as many straight lines passing through a given fixed point on the plane as there are circles with that fixed point as the centre ?

Problem 3

Suppose you have drawn 3 lines on a circle. What is the maximum number of regions that you can form ? Suppose you have drawn n lines on a circle. Can you find the maximum number of regions that can be formed ?

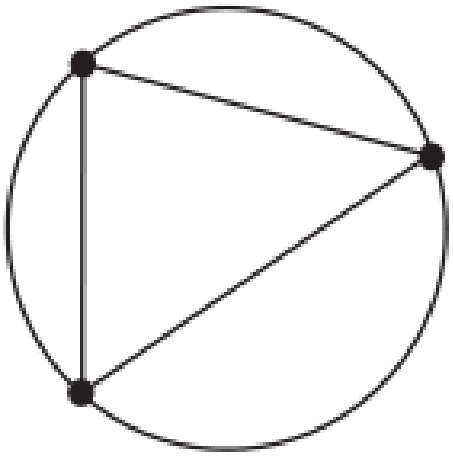

Figure 1: The case for $n = 3$ with 4 regions

Problem 4

Suppose a stick of length 1 has been broken into three parts. Can we always form a triangle with these 3 pieces ? What is the probability that we can form a triangle ?