Math-Circle: Session 3

TIFR-CAM and ICTS

November 21, 2022

π Again

Problem 1. *Let us consider convex figures in the plane. A convex figure has the property that if two points A and B are in the figure, then the straight line segment AB joining A and B is also in the figure. Define the diameter D of the figure as the largest possible distance between two points in the figure. Define the circumference C as the length of its perimeter. Show that C/D is always less than or equal to π. For what figure(s) is it equal to π? Is the answer unique?*

Note: This shows that π is the maximum possible ratio of C/D for convex figures. And, the circle is one of the maximising figure. In nature, you may encounter various symmetrical figures, such as spheres (shape of planets, soap bubbles etc..), or Romanesco broccoli (search it on google) and many others. Such symmetrical shapes appear naturally in nature often because such shapes minimise/maximise some quantity associated with the shape.

How to Measure

We are all familiar with the notion of length. For example, if A and B are two points, then the length of the line segment AB is obtained by putting a ruler between points A and B and reading the number given by the ruler. Note of course that there is an unit system used while taking this reading. For example it can be 15.24 cm is the SI unit, or it can be 6 inches in the imperial system (used in U.S.A). Therefore, depending on which unit system we use, we can assign different numbers to mean the length of the same object AB. This number is giving you a sense of how large your line segment is. Furthermore, apart from this discrepancy caused because of different unit systems, different people may as well have a different sense about how large the object is. For example, if a person has myopia or hypermetropia, his/her sense of size will be different from that of a person who has normal eyesight. Not only that, your sense of size may also depend on the direction you are looking at. For example, imagine you are wearing glasses which magnifies differently in different directions, i.e. the magnifying power of the glass is different in different portions of the glass.

Therefore, we understand that there is not a unique meaning to the "size" of an object. The size of an object can mean many many things. However, whenever we talk about the "size", it should satisfy some basic properties. For example, if AB and CD are two line segments which does not overlap with each other (i.e. their intersection is empty), then the total size of the union of AB and CD should be the sum of sizes of AB and CD.

In mathematics we have to reconcile all these discrepancies. Therefore, when we define the notion of size of some object, we do account for the fact that there can be many different meanings to the "size" of an object. But, we do impose the natural conditions it should satisfy. Having understood these explanations, let us now define the concept of a **measure**:

Let X be a set. For each subset $A \subset X$, we want to assign a number to A (which intuitively describes how large the set A is). Call this assigned number $\mu(A)$. Then, μ is called a measure if it satisfies the following property:

If $A_1, A_2, ...$ are pairwise disjoint sets, then

$$\mu(A_1 \cup A_2 \cup A_3 \cup ..) = \mu(A_1) + \mu(A_2) + \mu(A_3) + ... \tag{0.1}$$

Note that, apart from above restriction on μ, we have not put any other restriction. In particular, in this generalised notion of measure, the assigned number $\mu(A)$ can be any real number. It can even be a negative real number. Or, even more generally, it can also be a complex number.

Let us consider an example: Let $X = \mathbb{N} = \{1, 2, 3, 4, ...\}$ be the set of natural numbers. To each positive integer n, we assign a number a_n to it. We can then define a measure μ on $\mathbb{N}$ be assigning to each $A \subset \mathbb{N}$

$$\mu(A) = \sum_{n \in A} a_n.$$

Check that the above definition of μ satisfies the condition (0.1).

Use the above generalised notion of a measure to solve the following problems:

Problem 2. *Consider arithmetic progressions*

$$A_1 = \{a_1, a_1 + d_1, a_1 + 2d_1, ...\},$$

$$A_2 = \{a_2, a_2 + d_2, a_2 + 2d_2, ...\},$$

$$A_3 = \{a_3, a_3 + d_3, a_3 + 2d_3, ...\},$$

$$\vdots$$

$$A_m = \{a_m, a_m + d_m, a_m + 2d_m, ...\},$$

where for each A_k, a_k is the first term and d_k is the common difference. Assume that arithmetic progressions $A_1, A_2, ..A_m$ are pairwise disjoint. Also, assume that $\mathbb{N} = \cup_{k=1}^{m} A_k$. Prove that

- $\frac{1}{d_1} + \frac{1}{d_2} + .. + \frac{1}{d_m} = 1$
- *For some $i \neq j$, $d_i = d_j$.*

- *(Bonus problem) Can you find all such arithemetic progressions $A_1, A_2, ..A_m$ such that the above conditions hold?*

Hint: Given $\mathbb{N} = \cup_{k=1}^{m} A_k$ and A_k are pairwise disjoint, the first natural conclusion is that size of $\mathbb{N}$ must be equal to sum of sizes of A_k. If your notion of size of a set is the number of elements in that set (i.e. its cardinality), you will get $\infty = \infty + \infty + \infty + ... + \infty$. This does not give you any useful information. Recall however that you have a freedom in choosing the notion of size. This is done by choosing any numbers a_n of your choice as explained above. Note that the standard notion of size using cardinality corresponds to $a_n = 1$ for all n. Think of some other choice of numbers a_n so that sum $\sum_{n \in A_k} a_n$ can be computed in terms of a simple formula. Then, use the conclusion that the size of $\mathbb{N}$ must be equal to sum of sizes of A_k to derive information about arithmetic progressions A_k.

Problem 3. *Let R be a rectangle. Suppose you tile the rectangle R using smaller rectangles $R_1, R_2, .., R_m$. The sides of each rectangle R_k is parallel to the corresponding sides of R. The dimensions of rectangles R_k, i.e. lengths of its sides, may be different from each other (here, by length I mean the standard length in SI unit). Suppose that each rectangle R_k has at least one of its side (either its length or its breadth or both) of integer standard length in SI unit. Prove that the bigger rectangle R also has at least one of its side of integer standard length in SI unit.*
Follow the following steps to solve the above problem:

- *If $[a, b]$ is an interval on the real line, the standard notion of length gives the length of $[a, b]$ to be $b - a$. But, with the generalized notion of length as described above, for any function F (e.g. $F(x) = x^2, e^x, sin(x)$ or any other function of your choice), you can assign to interval $[a, b]$ the number $F(b) - F(a)$ as its length. That is, we are defining $\mu([a, b]) = F(b) - F(a)$. Check that this definition indeed defines a measure μ which satisfies (0.1).*

- *If R is a rectangle with its vertices at $(a, x), (b, x), (b, y), (a, y)$ in the coordinate plane, the standard notion of area gives the area of this rectangle as $(b - a)(y - x)$. But, with the generalized notion of area as described above, for any function F, you can assign to the rectangle R the number $(F(b) - F(a)) \times (F(y) - F(x))$ as its area. That is, we are defining $\mu(R) = (F(b) - F(a)) \times (F(y) - F(x))$. Check that this definition indeed defines a measure μ which satisfies (0.1).*

- *Now lets go back to the framework of the above problem. If a rectangle R is tiled using smaller rectangles $R_1, R_2, ..R_m$, the most immediate conclusion is that the area of R is sum of areas of R_k. That is,*

$$\mu(R) = \mu(R_1) + \mu(R_2) + .. + \mu(R_m). \tag{0.2}$$

  If you work with the standard notion of area which is $\mu(R) = (b-a)(y-x)$, the equation (0.2) may not give you any information. But, note that you also have a freedom of the function F while defining the notion of area. Try to think of a function F so that you can derive information about the rectangle R using the equation (0.2). While thinking of such a function F, try to use the given information that at least on side of each rectangle R_k is of integer length.