Maths Circle India

TIFR-STCS Maths Circle Team

Session 3: May 20, 2022

1 Paintings

It is summer vacation and Gopal is again visiting his uncle, aunt, and his little cousin, Jiya. In his previous vacation, he had made several friends in the neighbourhood, and now he goes to play cricket with them every evening. But now the vacations are about to end, and Gopal is yet to complete his vacation homework for drawing and painting. Sitting by the window with a couple of white sheets and water colors, Gopal gets the idea to paint a scenery with blue sky, and green grass, with faint and snowy mountains in the background. He has just taken out his blue and green paints in a tray, and water in a bowl, when his friends call him for the game. Excited, he rushes to play cricket leaving the paints and sheets open in the hall.

All this while Jiya had been sleeping quitely in the hall. But due to the slamming of the door, she gets up. She sees the colors lying on the floor. Intrigued, she dips her hands, one in blue, other in green, and randomly places her hands on all the sheets that Gopal had left behind and her face, coloring all of them completely with random patterns of green and blue.

On returning, Gopal notices the random patterns made by Jiya on his drawing sheets and makes a curious observation - "No matter how Jiya colored the sheets, I can always find at least 2 points, exactly 1 cm apart that are both either blue, or both green".

Can you help Gopal understand this observation?

2 Target chasing

During his summer vacations, Lava has developed a new *solitaire* game (i.e., a game that one can play alone, without any opponents) that goes as follows. He draws two long perpendicular line segments on a piece of paper, one horizontal, and the other vertical, which intersect at a point O. Let the right-most point on the horizontal line segment be A. He then draws an arrow, called *target* of length 5cm with its tail at O, and calls its head T. Let the angle AOT be called ϕ.

He the draws another, called *chaser* of length 5cm with its tail at O, and calls its head C. Let the angle AOC be called θ.

In each move of the game, only the chaser is moved: the target remains fixed in place. Each such move consists of two moves:

1. First, reflect the chaser in the horizontal line.
2. Then, reflect the chaser (in its new position) in the target line.

So, how does the game progress when the chaser is initially 70° "below" the horizontal line (so $\theta = -70°$), and the target is 10° above the horizontal line (so $\phi = 10°$)? How about when the initial positions are $-70°$ and 5° respectively? Can you give a general "rule" for figuring out what will happen?

3 Tilings

Arun's friend Barun has moved into a new house and he invites Arun for the housewarming party. But Barun tells him that they're facing a problem. His room still needs to be tiled. "The floor my room is an 8×8 square. Weirdly enough, my parents have already purchased L-shaped tiles, each tile being made of three 1×1 square tiles. Barring *this* single square unit on the floor, where we'll have a huge Beyblade stadium installed, we need to tile the room's floor with these L-shaped tiles."

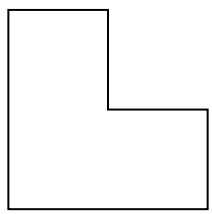

Figure 1: The L-shaped tile made of three 1×1 square tiles

Can Arun help Barun tile his 8× 8 room using L-shaped tiles while ensuring that the single square (specified by Barun) remains un-tiled on the floor? Or is this impossible? What about a smaller 4×4 room? Or a larger 16×16 room?

4 Rhyme time

Jai is reading a book of funny poems. He finds a four-line poem he likes particularly, and reads it out to his little sister Riya.

> Oh, if you're a bird, be an early bird
> And catch the worm for your breakfast plate.
> If you're a bird, be an early early bird—
> But if you're a worm, sleep late.
>
> — Shel Silverstein

Riya notices that the first and third lines rhyme, and the second and fourth lines rhyme. But she also knows other four-line poems which have a different pattern. In "Twinkle Twinkle Little Star", the first and second lines rhyme, and the third and fourth lines rhyme.

Jai wonders if he can count how many such patterns are there. 'In Twinkle Twinkle Little Star, suppose we call lines ending in words that rhyme with "star" as type A, and words that rhyme with "high" as type B. Then it has an AABB pattern. In the other poem, the types are different, but you could say the pattern is ABAB.'

'Oh!' says Riya. 'Then there are two possibilities for each line, A or B, and so the total number is 2^4, or 16.'

'No, that can't be right.' says Jai. 'Then we are counting AAAA and BBBB separately. But in both, all four lines are the same type. That's the same pattern. I want to count each pattern only once.'

Can you help out Jai?

What if there are three types? For instance, some four-line poems have a pattern ABCB.

> Said Hamlet to Ophelia,
> I'll draw a sketch of thee,
> What kind of pencil shall I use?
> 2B or not 2B?
>
> — Spike Milligan

How many total patterns are there for four-line poems (for any number of types)? What about five-line or six-line poems? Can you come up with a general way of getting this number?