Nonergodic extended states and many body localization proximity effect through real-space and Fock-space excitations

Sumilan Banerjee

Centre for Condensed Matter Theory, Department of Physics,
Indian Institute of Science

Stability of Quantum Matter, ICTS
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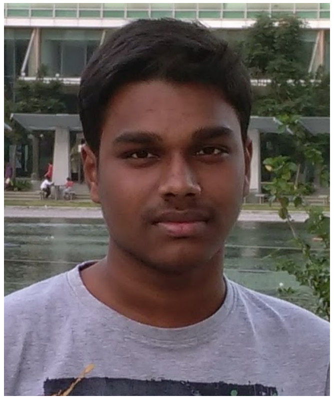
Nilanjan Roy
(NTU)



Jagannath Sutradhar
(IISc → Bar Ilan)

Soumi Ghosh (ICTS), Sthitadhi Roy (ICTS), David Logan (Oxford),
Subroto Mukerjee (IISc)

N. Roy, J. Sutradhar & SB, PRB, 107, 115155 (2023).

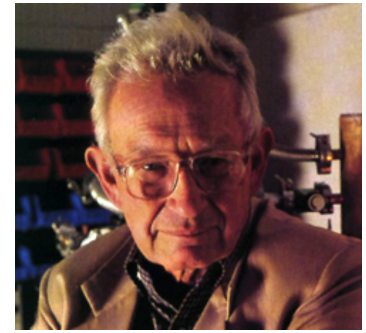
J. Sutradhar, S. Ghosh, S. Roy, D. Logan, S. Mukerjee & SB, PRB 106, 054203 (2022).

S. Ghosh, J. Sutradhar, S. Mukerjee, SB, arXiv:2401.03027.

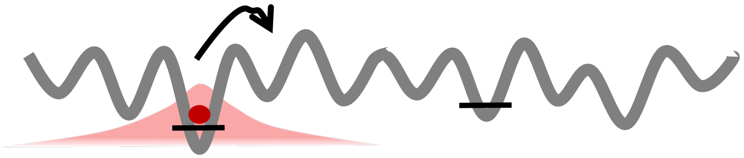
Anderson localization

Anderson (single-particle) localization (1958)

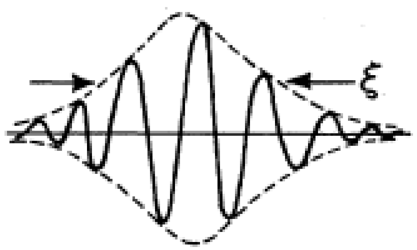
Abrahams et al. (1979), Lee & Ramakrishnan (1985), ...



Isolated system

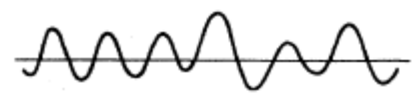


Localized



$$|\psi_\alpha(r_i)|^2 \sim \frac{1}{\xi^d} e^{-\frac{|r_i - r_\alpha|}{\xi}}$$

Extended

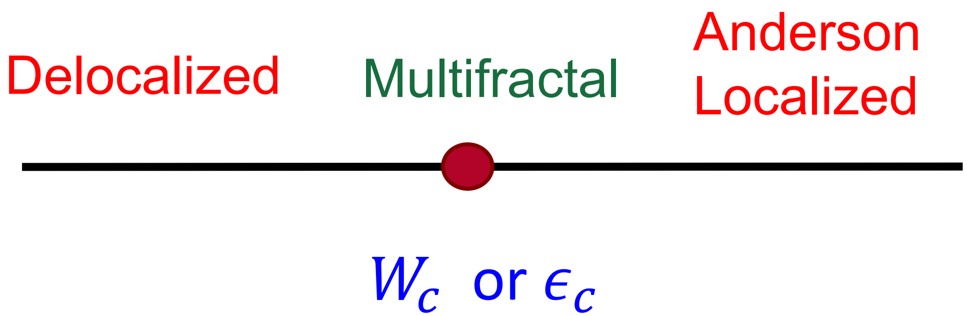


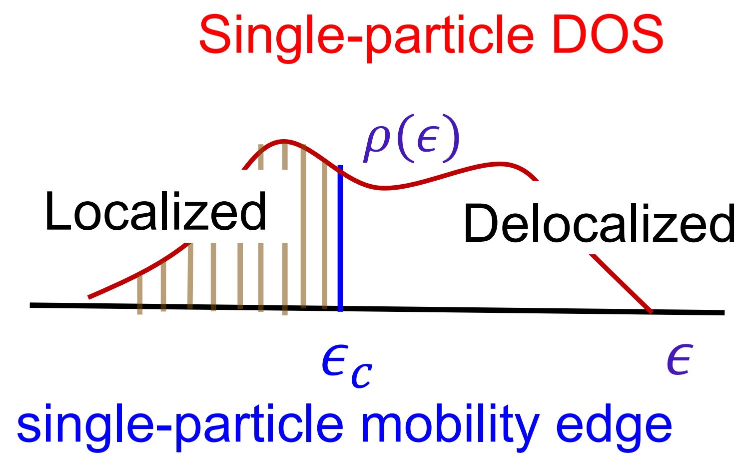
$$|\psi_\alpha(r_i)|^2 \sim \frac{1}{L^d}$$

$$\mathcal{H} = -t \sum_{<ij>} (c_i^\dagger c_j + h.c.) - \sum_i \varepsilon_i n_i$$

$$\varepsilon_i \in [-W, W]$$

Usual Anderson transition (e.g. $D = 3$)   IPR in real space

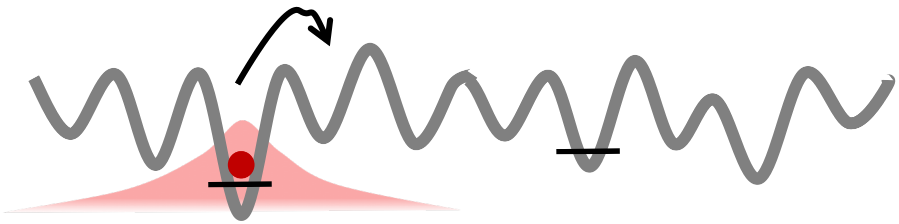




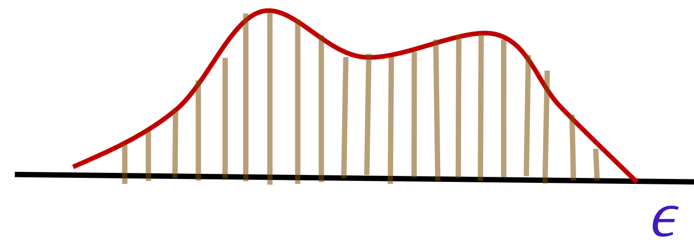
Generalized IPR
$$I_q = \sum_{i=1}^{N} |\langle i|\psi_\alpha\rangle|^{2q} \sim N^{-D_q}$$

- Delocalized states $D_q = q - 1$, $I_2 \sim N^{-1}$
  $\Rightarrow$ spread over all sites
- Localized states $D_q = 0$ $I_2 \sim \mathcal{O}(1)$
  $\Rightarrow$ spread over finite number of sites
- Fractal or multifractal
  non-trivial D_q, $0 < D_q < 1$, $I_2 \sim N^{-D_2}$
  $\Rightarrow$ spread over N^{D_2}, but zero fraction of the sites

Many-body localization (MBL)



Start with all states localized



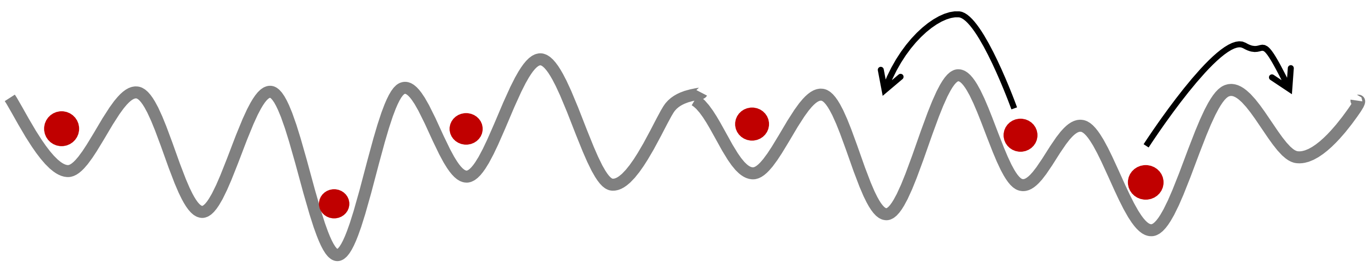
$$\mathcal{H} = -t \sum_{<ij>} \left(c_i^\dagger c_j + h.c.\right) - \sum_i \varepsilon_i n_i \quad + V \sum_{\langle ij \rangle} n_i n_j$$

$$|\psi_\alpha(r)|^2 \sim \frac{e^{-\frac{|r - r_\alpha|}{\xi}}}{\xi^d}$$

$$= \sum_\alpha \epsilon_\alpha c_\alpha^\dagger c_\alpha \quad + \sum_{\alpha\beta\gamma\delta} V_{\alpha\beta\gamma\delta} c_\alpha^\dagger c_\beta^\dagger c_\gamma c_\delta$$

$$\varepsilon_i \in [-W, W]$$

Add interaction



At high energies interaction connects between $\sim \exp(N)$ localized states ! Can localization survive?

Many-body localization (MBL)

Yes!
For sufficiently strong disorder

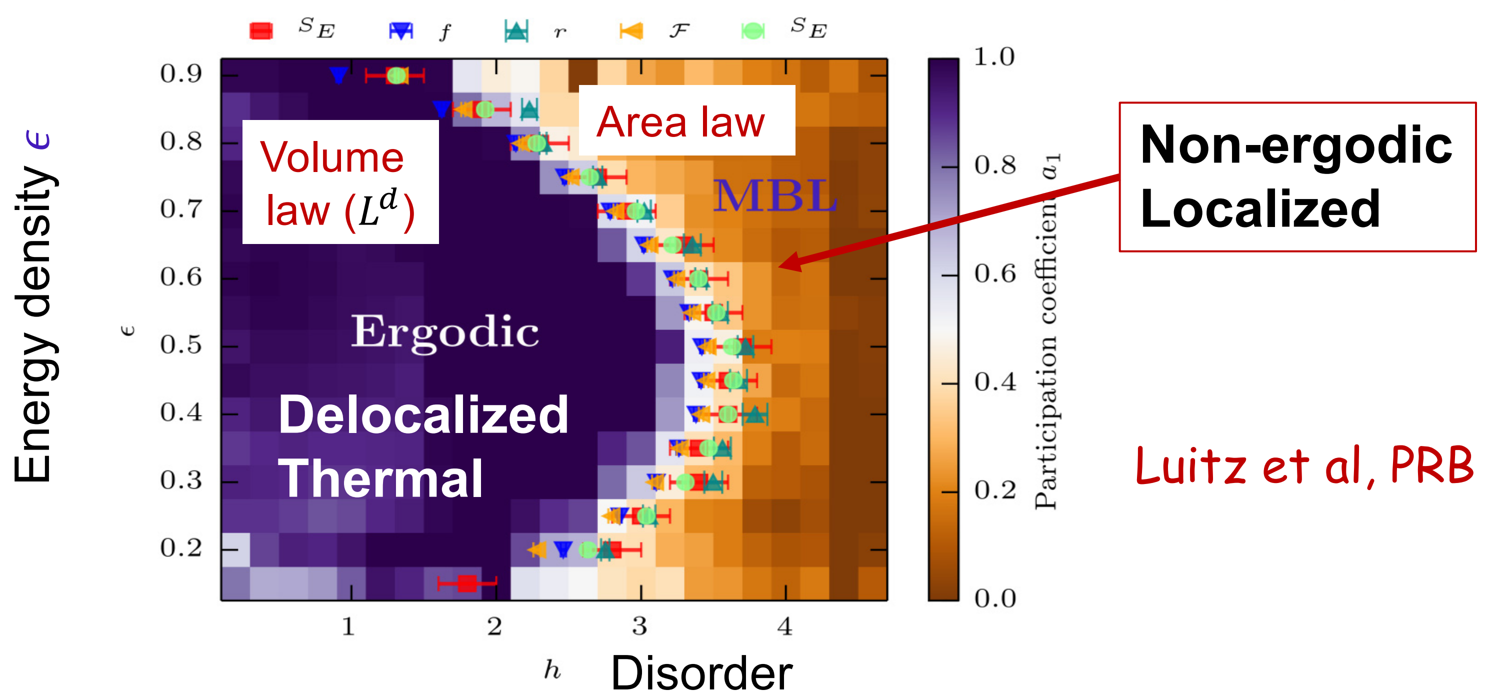
Basko, Aleiner, Altshuler (2005);
Gornyi, Mirlin, Polyakov (2005)

* Mathematical proof with mild assumption in a particular model
Imbrie, J. Stat. Phys. (2016), PRL (2016)

Perturbative treatment for weak interaction,
$V/\Delta_\xi \ll 1$ in dimension $d > 1$
$\Delta_\xi \sim 1/N(0)\xi^d$, single-particle level spacing within a localization volume

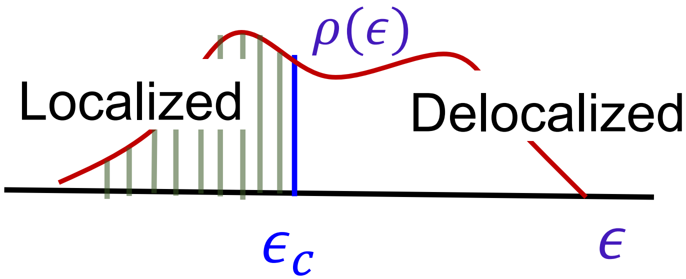
(Finite-size) MBL

Oganesyan and Huse (2007),
Pal and Huse (2010), ….



Luitz et al, PRB (2015)

What happens we start with a mobility edge (or delocalized single-particle (SP) states)?



System with protected delocalized SP state(s)

→ Delocalization of all SP states

Nandkishore & Potter, PRB (2015); Potter & Vasseur, PRB (2016); SB & Altman, PRL (2016); ....

Generalized Aubry-Andre-Harper (GAAH) models   Ganeshan et al., PRL (2015)

→ MBL can exist even with mobility edge

Ganeshan et al., PRL (2015);
Modak & Mukerjee, PRL (2015);
Nag and Garg, PRB (2017);
Ghosh et al., PRB (2020);......
*Pomata et al. (2020)

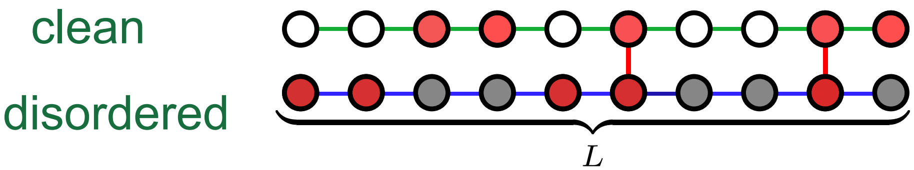
→ States intermediate between MBL and ergodic

Non-ergodic extended (NEE) states

Phases	Entanglement entropy	Subsystem number fluctuations	Level spacing statistics
MBL	Area law	Finite for $L \to \infty$	Poisson
NEE	Volume law	Finite for $L \to \infty$	Intermediate between Poisson and GOE
Ergodic	Volume law	Vanishes for $L \to \infty$	GOE

How non-ergodic states are realized with SP mobility edge?

MBL proximity effect   Nandkishore, PRB (2015)

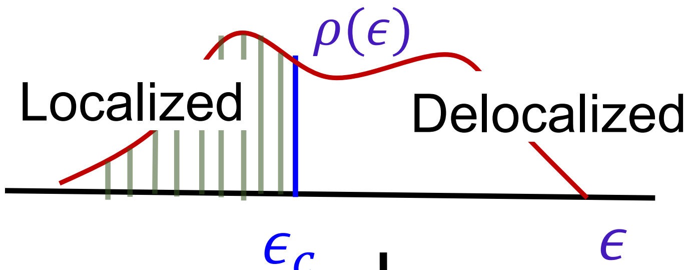


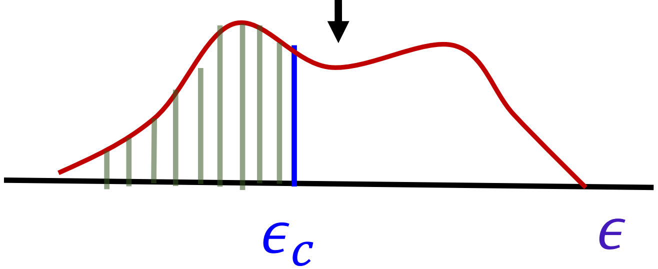
Delocalized SP states get localized by coupling with localized states

How to see in presence of interaction?

Hyatt et al., PRB (2017), Marino et al., PRB (2018)

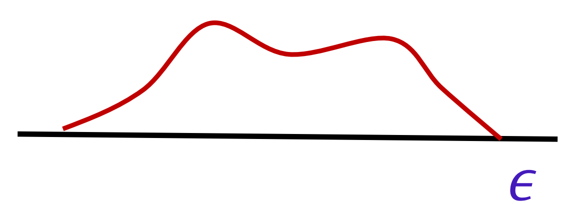
Non-interacting SP local DOS



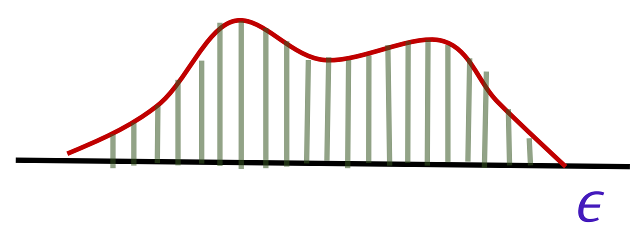


Mobility edge in SP excitations
→ NEE states

Local SP excitations with interaction



All SP excitations delocalized
→ Delocalized many-body state

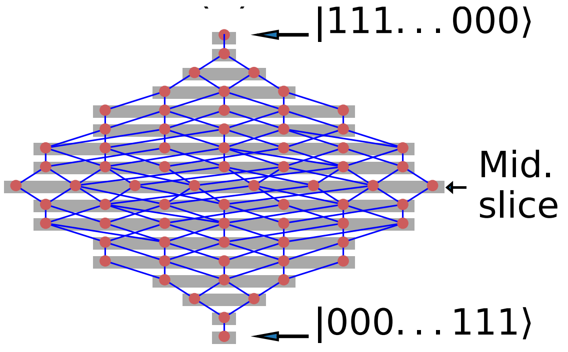


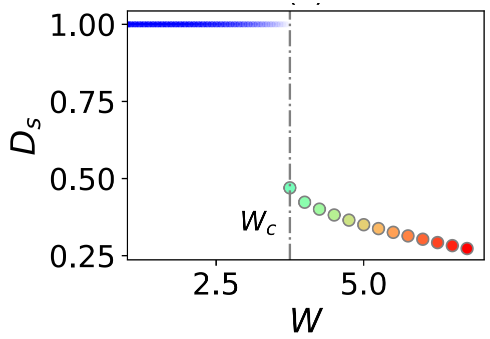
All SP excitations localized
→ MBL (proximity effect)

Plans for rest of the talk

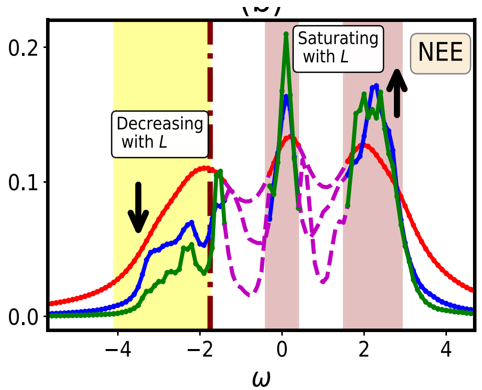
○ Real-space and Fock space (FS) pictures for MBL and MBL transition with random disorder

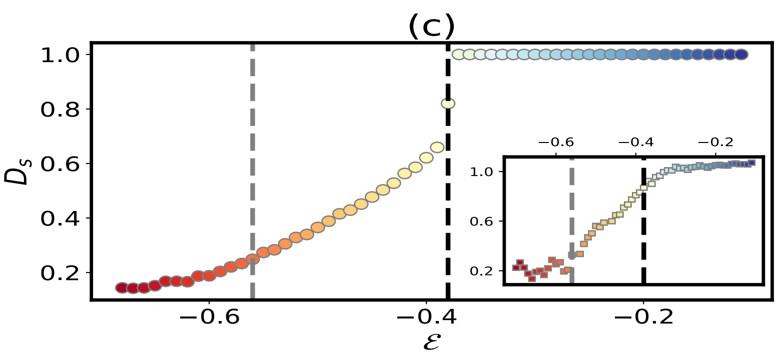
Recursive Green's function
FS propagator
FS multifractality and scaling theory across the transition

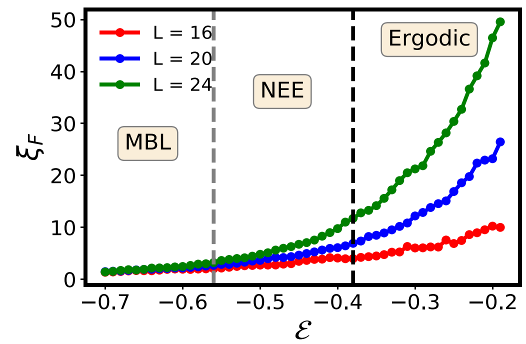




○ MBL proximity effect and NEE states in a generalized Aubry-Andre-Harper (GAAH) model.

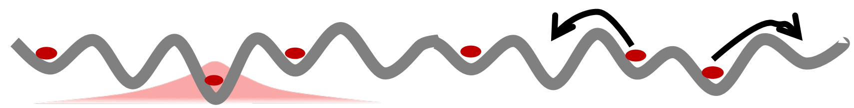






○ Conclusions

MBL: Real-space picture



$$H = \sum_\alpha \epsilon_\alpha c_\alpha^\dagger c_\alpha + \sum_{\alpha\beta\gamma\delta} V_{\alpha\beta\gamma\delta} c_\alpha^\dagger c_\beta^\dagger c_\gamma c_\delta$$

Vosk & Altman, PRL (2013)
Serbyn et al., PRL (2013)
Huse & Oganesyan, PRB (2013)
Chandran et al., PRB (2015);...

MBL fixed point → Emergent integrability
→ Quasi local integrals of motion
(LIOMs or "l-bits")

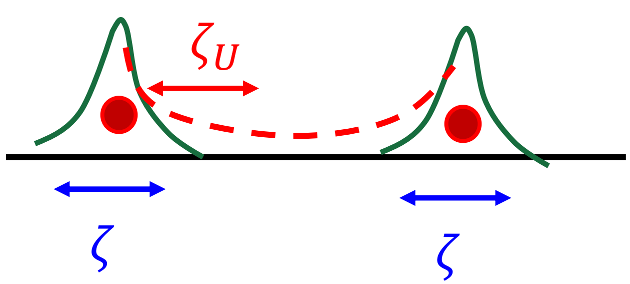
$$\tilde{c}_\alpha^\dagger = \sum_i A_\alpha(i) c_i^\dagger + \cdots$$

$$\tilde{n}_\alpha = \tilde{c}_\alpha^\dagger \tilde{c}_\alpha$$

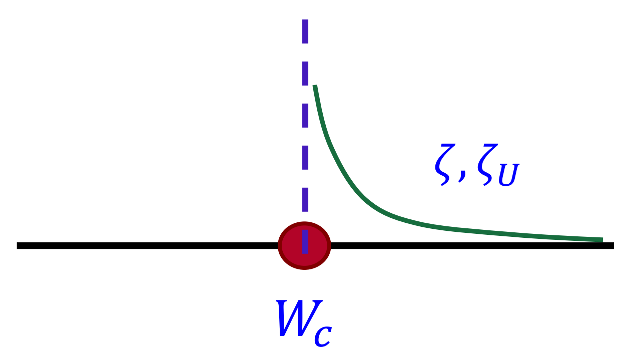
$$A_\alpha(i) \sim \exp[-\frac{|i - i_\alpha|}{\zeta}]$$

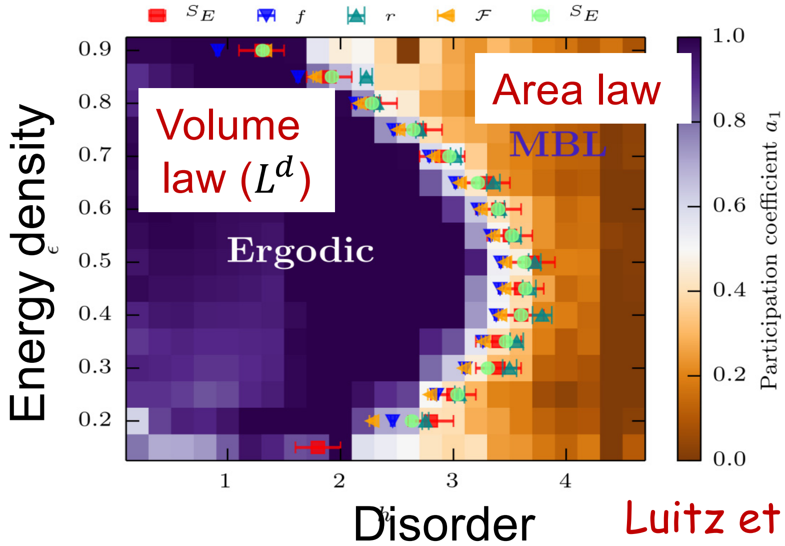
$$H_l = \sum_\alpha \epsilon_\alpha \tilde{n}_\alpha + \sum_{\alpha\beta} U_{\alpha\beta}\, \tilde{n}_\alpha \tilde{n}_\beta + \cdots$$

$$U_{\alpha\beta} \sim \exp\left[-\frac{|i_\alpha - i_\beta|}{\zeta_U}\right]$$



Do $\zeta, \zeta_U \to \infty$
for $W \to W_c^+$?



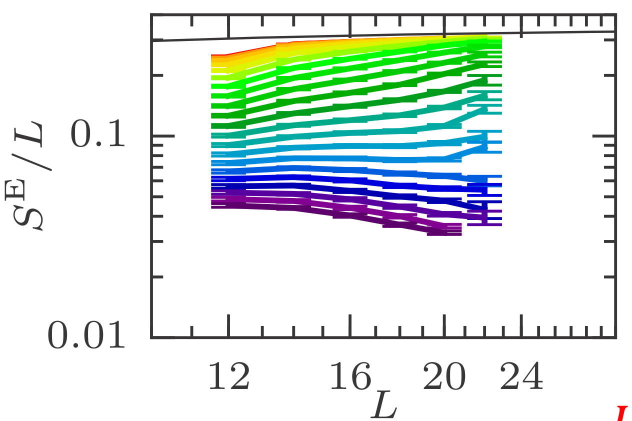


Luitz et al, PRB (2015)

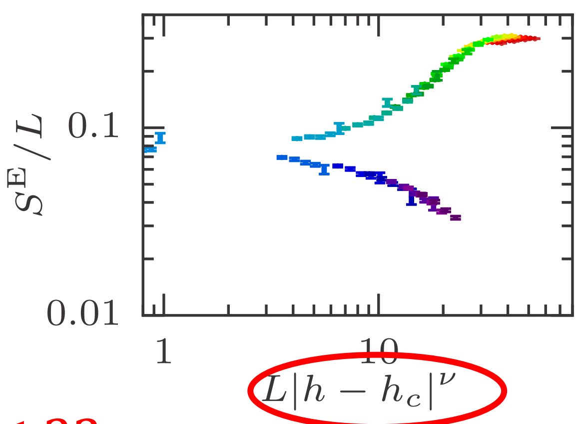
What is the nature of the transition? First order, continuous, ..?

Is there diverging length scale? Critical properties, exponents, ..?

Theory of the entanglement transition?



$L \leq 22$

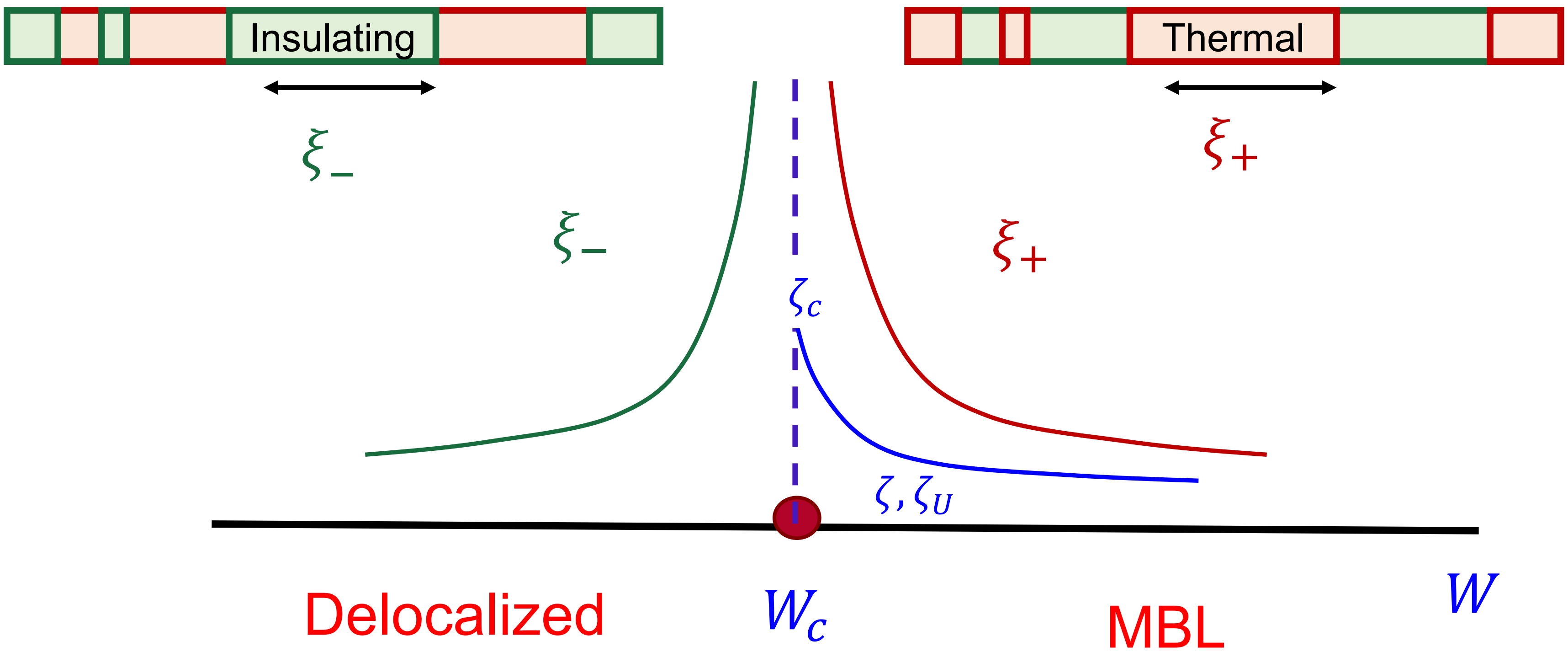


Correlation length exponent $\nu \leq 1$

Violates Harris-CCFS bound

$\nu \geq 2/d$

Real-space picture



Phenomenological renormalization groups (RG)

$$\xi_\pm \sim |W - W_c|^{-\nu}$$

Potter et al PRL (2015),
Vosk et al PRL (2015),

Continuous transition

$$\sim \exp\left[\frac{b_\pm}{\sqrt{|W - W_c|}}\right]$$

Thiery et al PRL (2018),
Goremykina et al PRL (2019),
Dumitrescu et al, PRB (2019)

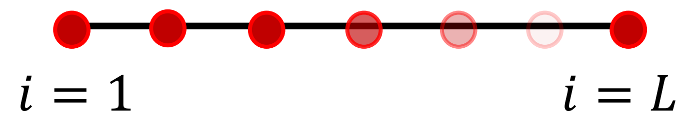
KT transition

MBL through quantities related to Fock space

$$\mathcal{H} = t\sum_{i=1}^{L-1}(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) + \sum_{i=1}^{L}\epsilon_i n_i + V\sum_{i=1}^{L-1} n_i n_{i+1}$$

L sites, N fermions $\quad \sum_i \hat{n}_i = \hat{N}$ conserved

Real space



$L = 4, N = 2$

$|1100\rangle$
$|1010\rangle$
$|0110\rangle$
$|1001\rangle$
$|0101\rangle$
$|0011\rangle$

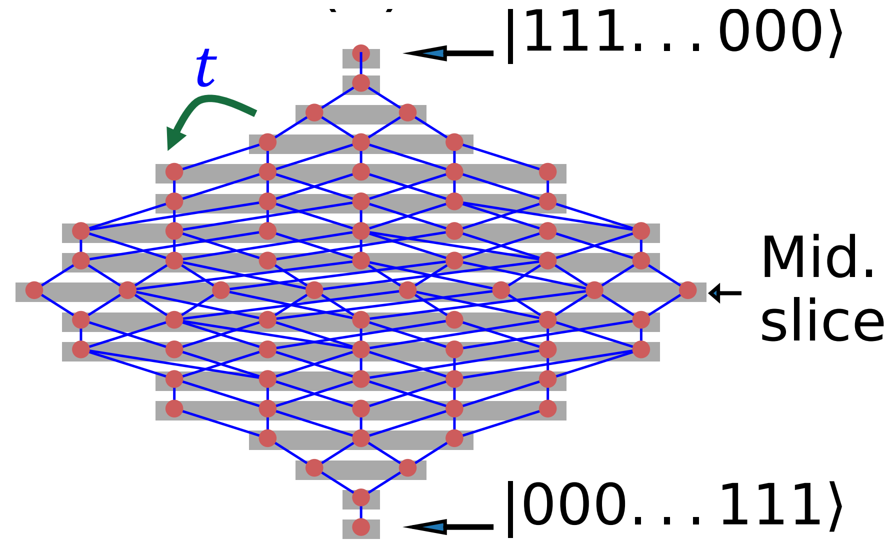
Number of FS sites

$$N_F = \frac{L!}{N!\,(L-N)!}$$

$\sim \exp(L)$ (Large L)

Fock space (FS) lattice

Logan and Welsh, PRB (2019)



FS basis sites $|I\rangle = |n_1^{(I)} n_2^{(I)} \dots n_L^{(I)}\rangle$

Tight binding model on the FS lattice

$$\mathcal{H} = \sum_{I \neq J} T_{IJ}|I\rangle\langle J| + \sum_I \mathcal{E}_I |I\rangle\langle I|$$

ϵ_i, L random numbers $\Rightarrow \exp(L)$ numbers

$$\mathcal{E}_I = \sum_i \epsilon_i n_i^{(I)} + V\sum_i n_i^{(I)} n_{i+1}^{(I)}$$

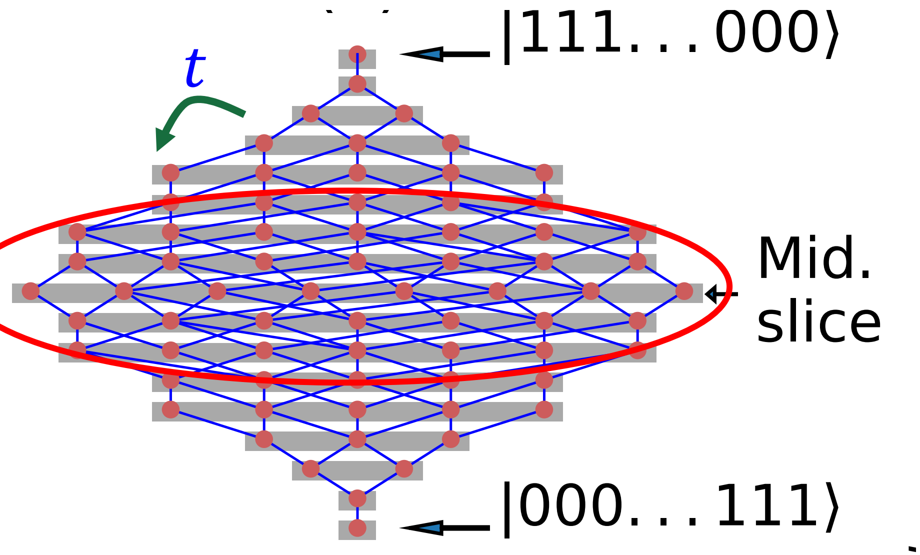
Extremely correlated disorder in Fock space

MBL $\equiv$ Anderson localization on a complex lattice with correlated disorder

Coordination number of the FS lattice $\sim L$ diverges for $L \to \infty$

$\Rightarrow$ Requires correlated disorder for localization

Altland & Micklitz, PRL (2017)
Logan and Welsh, PRB (2019);
Ghosh et al., PRB (2019)
Roy & Logan, PRL (2020)



Many-body eigenstate $|\Psi_n\rangle = \sum_I C_{nI} |I\rangle$

How to detect localization on FS lattice?

Inverse participation ratio (IPR) $I_q = \sum_I C_{nI}^{2q}$ Usual IPR, I_2

or, participation entropy $S_q = \frac{1}{1-q} \ln(I_q)$

$$I_q \sim N_F^{-D_q}$$

- Delocalized states $D_q = q - 1,\ \ S_q = \ln N_F$ $\qquad I_2 \sim N_F^{-1}$

  $\Rightarrow$ spread over all FS sites

- Localized states $D_q = 0,\ S_q = \text{constant}$ $\qquad I_2 \sim \mathcal{O}(1)$

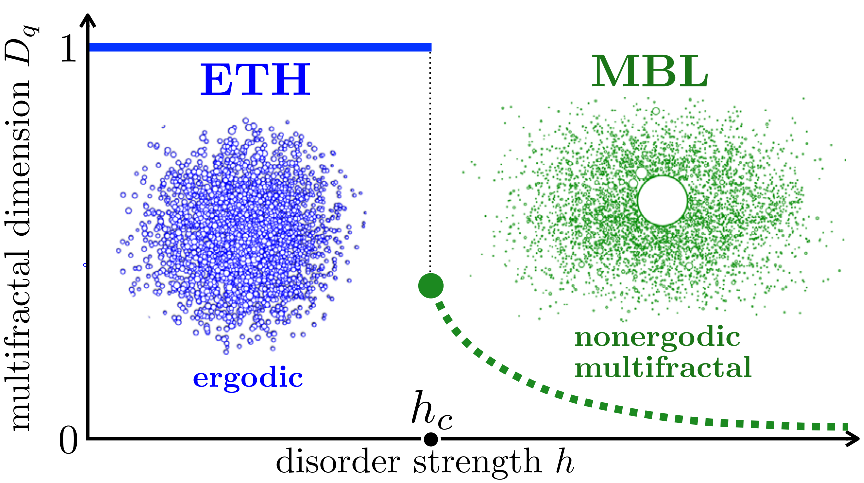
  $\Rightarrow$ spread over finite number of sites

- Fractal or multifractal

  non-trivial $D_q,\ 0 < D_q < 1,\ \ S_q = D_q \ln N_F$ $\qquad I_2 \sim N_F^{-D_2}$

  $\Rightarrow$ spread over $\exp(L)$, but zero fraction of the sites

What happens to MBL?



All MBL states are multifractal!

Mace et al., PRL (2019)

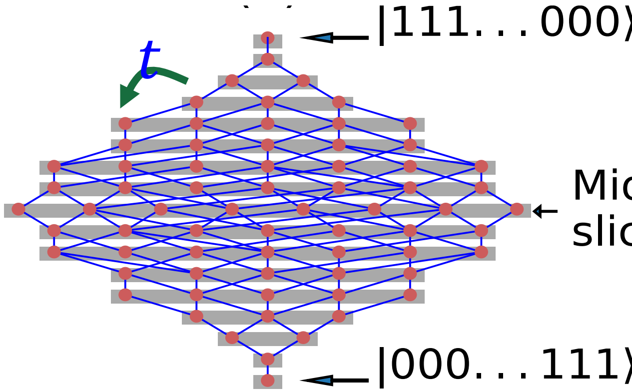
How does the multifractal nature manifest in dynamical properties?

Propagator or Green's function in Fock space

Tight binding model on the FS lattice

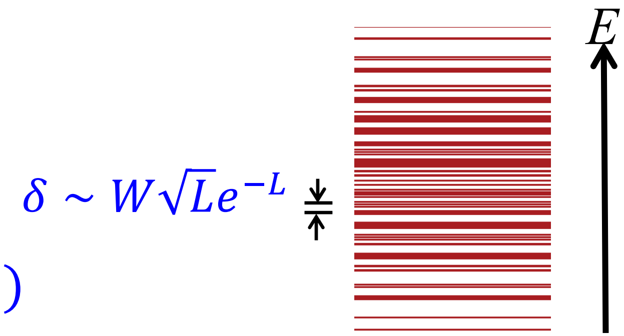
$$\mathcal{H} = \sum_{I \neq J} T_{IJ} |I\rangle\langle J| + \sum_{I} \mathcal{E}_I |I\rangle\langle I|$$

$$\hat{G}(E) = (E + i\eta - \mathcal{H})^{-1} \quad G_{IJ}(E) = \langle I|\hat{G}(E)|J\rangle$$



η, broadening parameter

We choose $\eta = \delta \sim \exp(-L)$



○ Diagonal part $[G_{II}(E)]^{-1} = E + i\eta - \mathcal{E}_I - \Sigma_I(E)$

Feenberg self-energy $\Sigma_I(E) = X_I(E) - i\Delta_I(E)$

$\Delta_I(E)$, imaginary part ⇒ Decay rate of FS site localized state at energy E

○ Off diagonal part $G_{IJ}(E) = G(r_{IJ}) \sim \exp[-r_{IJ}/\xi_F]$

r_{IJ}, distance on FS lattice, hopping distance

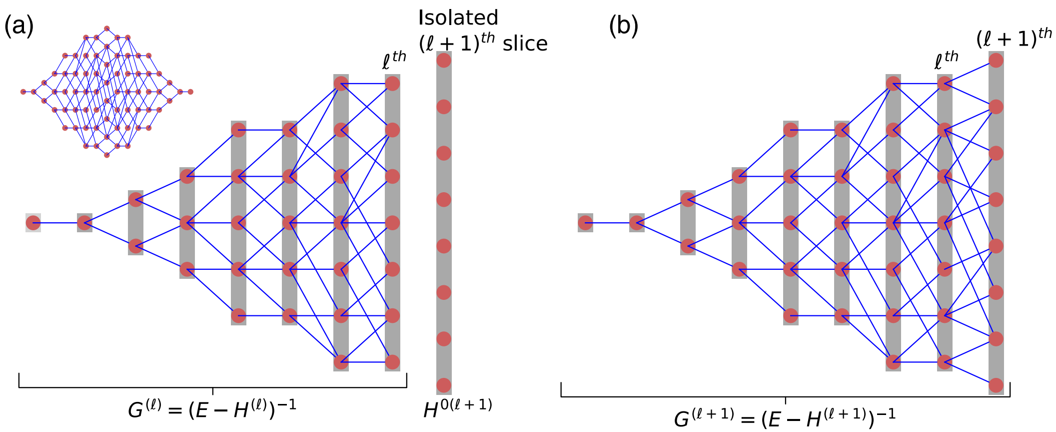
ξ_F ⇒ Fock space localization length

Recursive Green's function method

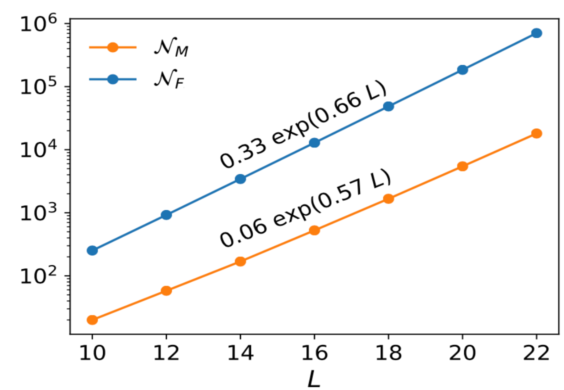
$$\mathcal{H} = \sum_{I \neq J} T_{IJ} |I\rangle\langle J| + \sum_{I} \mathcal{E}_I |I\rangle\langle I|$$

$$\hat{G}(E) = (E + i\eta - \mathcal{H})^{-1}$$

$$G_{IJ}(E) = \langle I|\hat{G}(E)|J\rangle$$



Recognition of the "slice structure"
⇒ well-known Recursive Green's function method for real space   MacKinnon & Kramer (~1980)



Relatively easy for $L \leq 22$

comparable to state-of-the-art ED methods $L \leq 22 - 24$

We calculate typical values and distributions of $\Delta_I(E)$ and $G_{IJ}(E)$

$$\Delta_t(E) = \exp(< \ln \Delta_I(E) >_{I,\{\epsilon_I\}})$$

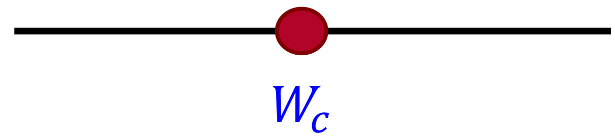
$$G(r_{IJ}) = \exp(< \ln G_{IJ}(E) >_{\{\epsilon_I\}})$$

*Geometric mean

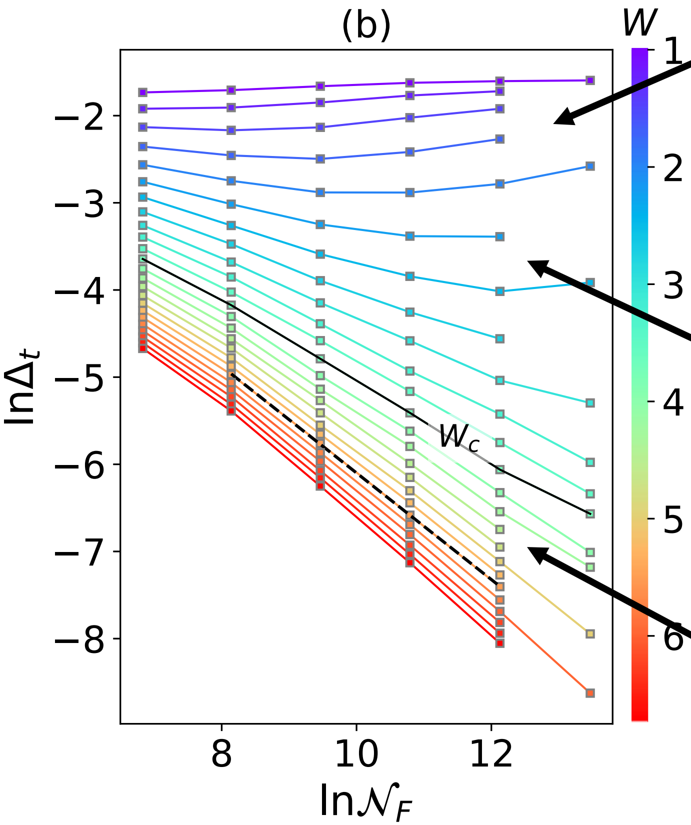
How do we detect MBL from Δ_t?

- Thermal phase $\Delta_t \sim \mathcal{O}(1)$ for $N_F \to \infty$, decay in finite time

- MBL phase $\Delta_t \to 0$ for $N_F \to \infty$,
infinitely long lived, memory of initial state

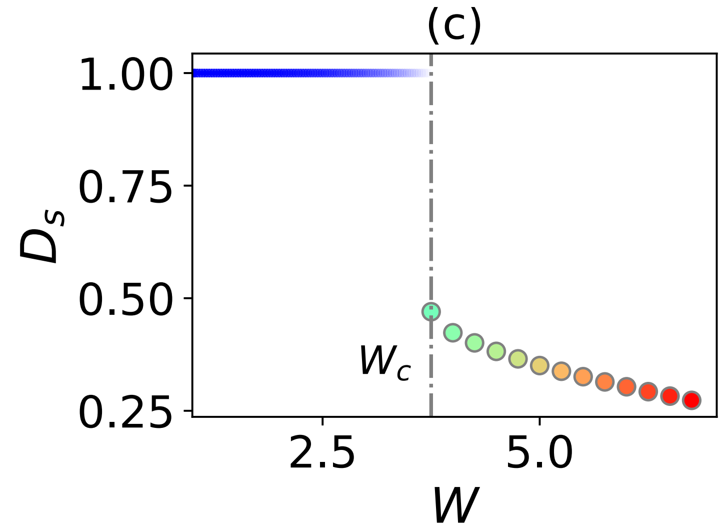


⇒ Like an order parameter for transition (Anderson 1958)



$\Delta_t \sim \mathcal{O}(1)$, Deep in the egodic phase

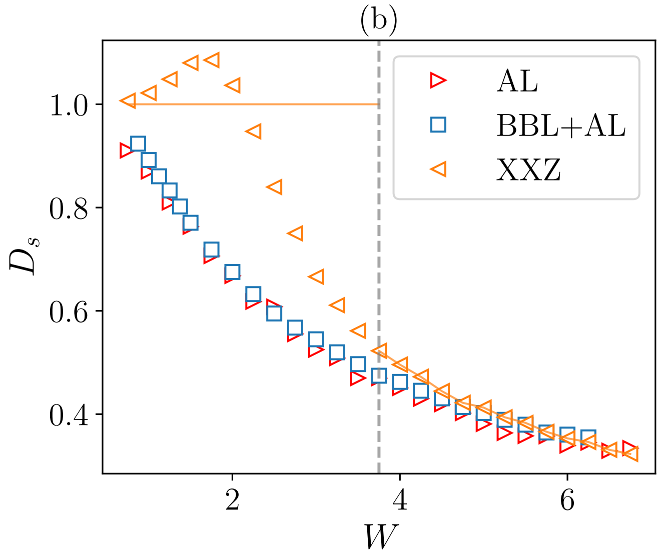
“Critical region”



$\Delta_t \sim N_F^{-(1-D_s)}$, Deep in the MBL phase
$D_s \simeq D_2$, spectral fractal dimension
$I_2 \sim N_F^{-D_2}$

Δ_t ⇒ MBL states are multifractal , D_s spectral fractal dimension

How does single-particle Anderson localization (AL) look in Fock-space?



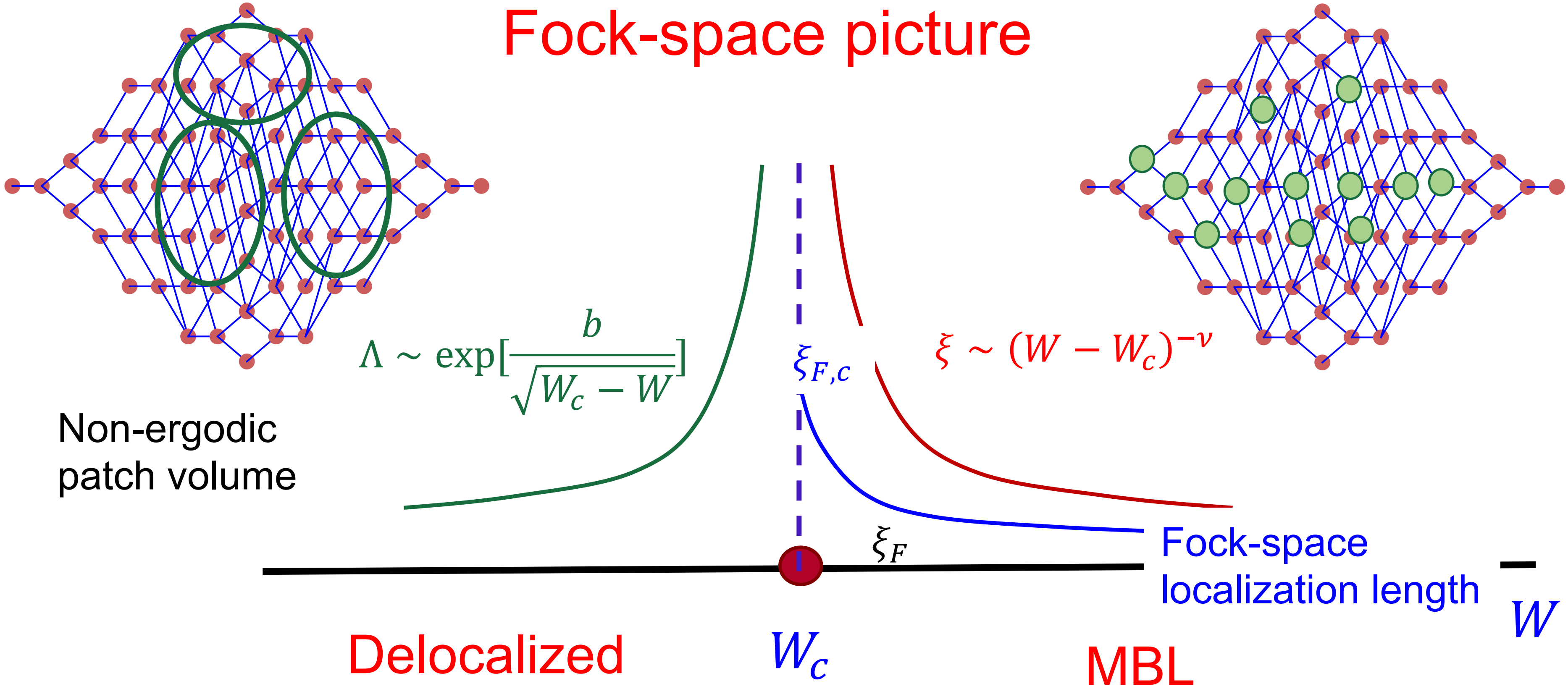
AL states also look multifractal in Fock space.

Real-space uncorrelated disorder

⇒ Correlated disorder in Fock-space

Multifractality is inevitable! D. E. Logan (unpublished)

Fock-space picture



Fock-space localization length $\xi_F(\leftarrow \; G(r_{IJ}))$ remains finite at the transition

J. Sutradhar, S. Ghosh, S. Mukerjee, SB (unpublished)

S. Ghosh, J. Sutradhar, S. Mukerjee, SB, arXiv:2401.03027

System with single-particle mobility edge

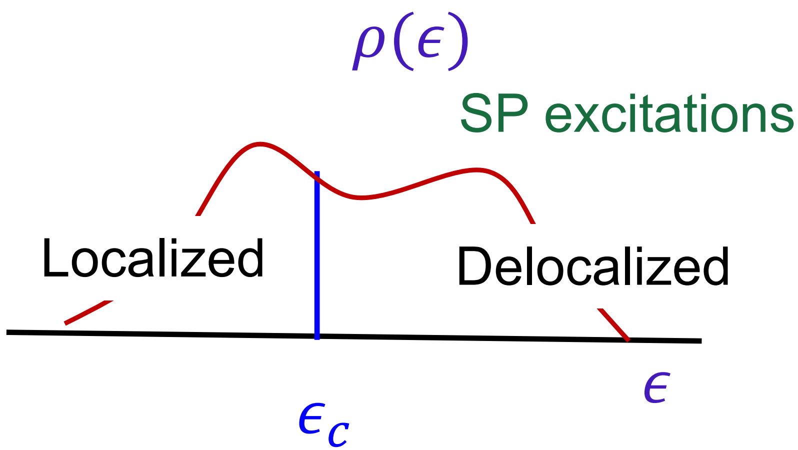
$$\mathcal{H} = t\sum_{i=1}^{L-1}(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i) + \sum_{i=1}^{L}\epsilon_i n_i + V\sum_{i=1}^{L-1} n_i n_{i+1}$$

Generalized Aubry-Andre-Harper (GAAH) potential Ganeshan et al, PRL (2015)

$$\epsilon_i = \frac{h\cos(2\pi b i + \phi)}{1 - \alpha\cos(2\pi b i + \phi)} \quad \alpha = -0.8$$

Exact mobility edge $\epsilon_c = \mathrm{sgn}(h)(2|t| - |h|)/\alpha$

Single-particle (SP) density of states (DOS)



single-particle mobility edge

What happens in the presence of interaction?

Quarter filling $N = L/4$

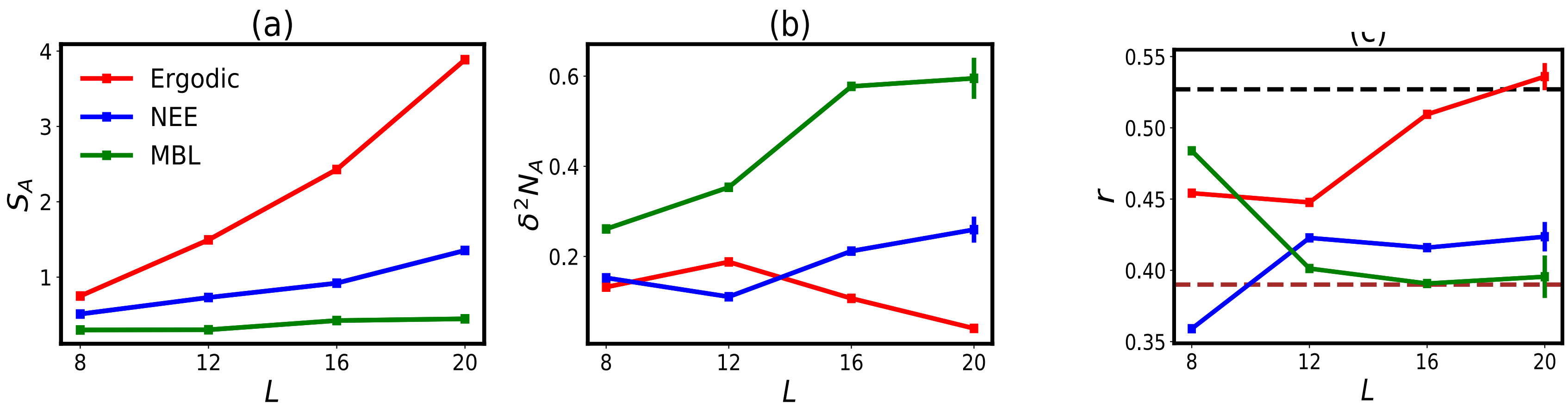
As a function of energy density $\mathcal{E} = E/L$ and $h = 0.6, 1.8$

$V = 1,\ L = 8 - 24$

Evidences of NEE in interacting GAAHS model

States in between ergodic delocalized and MBL

Li et al, PRL (2015); Modak et al., PRL (2015), ...

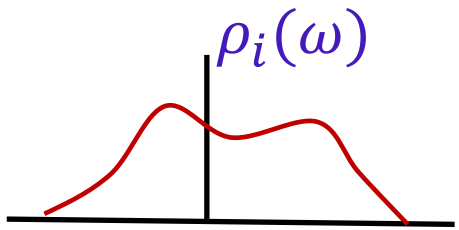


Phases	Entanglement entropy	Subsystem number fluctuations	Level spacing statistics
MBL	Area law	Finite for $L \to \infty$	Poisson
NEE	Volume law	Finite for $L \to \infty$	Intermediate between Poisson and GOE
Ergodic	Volume law	Vanishes for $L \to \infty$	GOE

How to detect localized/delocalized SP excitations?

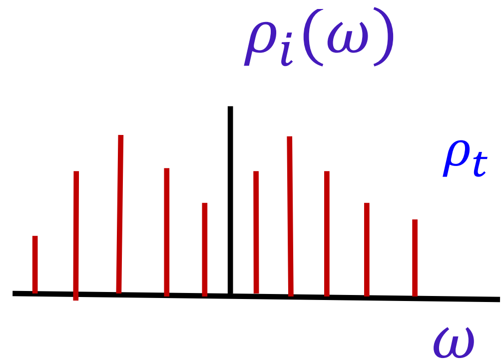
Local DOS (LDOS)

Delocalized



$\rho_t \simeq \rho_a \neq 0$

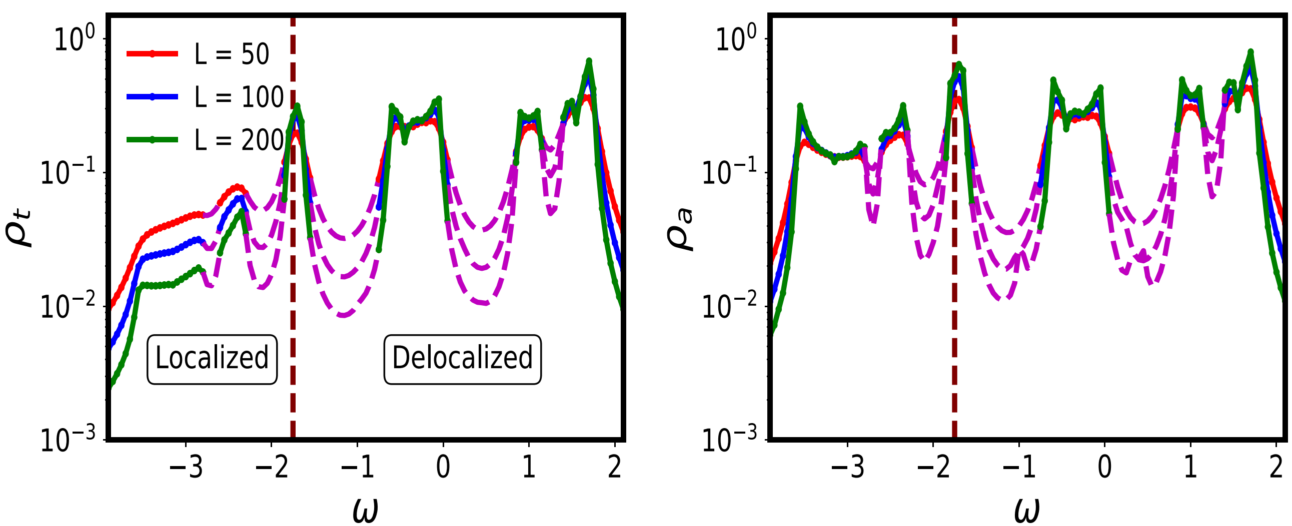
Localized



$\rho_t = 0$ for $L \to \infty$

$\rho_t \neq \rho_a \neq 0$

Non-interacting GAAH model ($V = 0$)



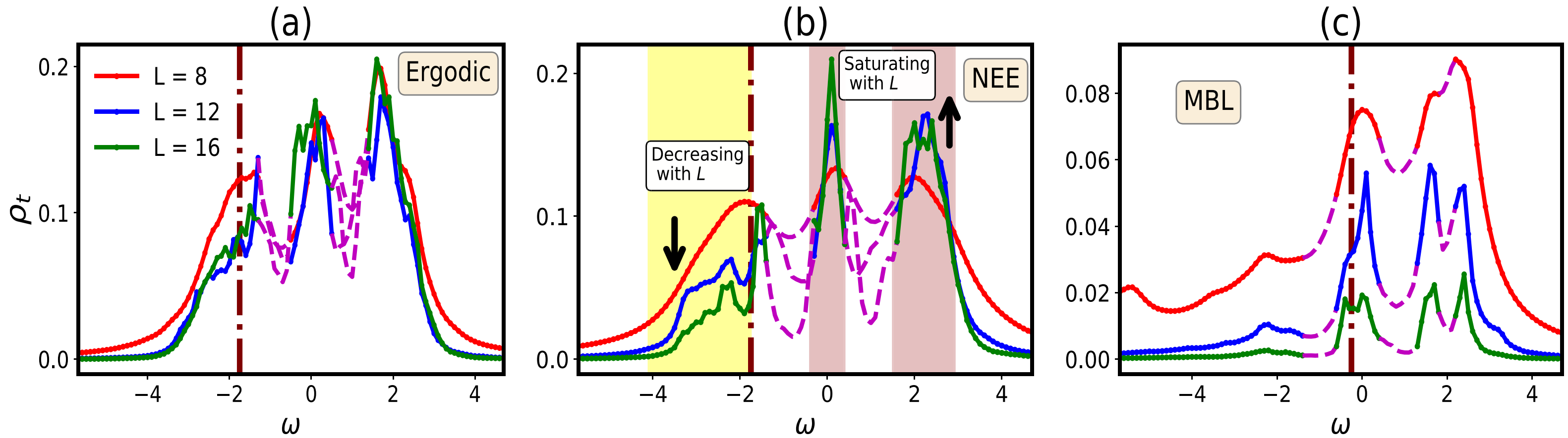
$$\rho_i(\omega) = \frac{1}{\pi} \sum_{\alpha=1}^{L} |\psi_\alpha(i)|^2 \frac{\eta_s}{(\omega - \epsilon_\alpha)^2 + \eta_s^2}$$

For spectral gaps

$\rho_t(\omega), \rho_a(\omega) \to 0$ as $L \to \infty$

Mobility edge in SP excitations persists in the NEE phase, unlike ergodic and MBL phases

Small system exact diagonalization (ED)



MBL proximity effect

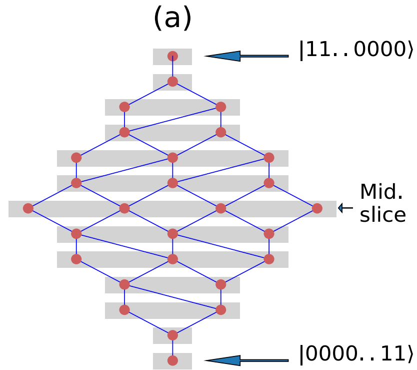
- Local spectral function of an eigenstate

$$\rho_{i,n}(\omega) = -\frac{1}{\pi}\mathrm{Im}G^R_{i,n}(\omega)$$

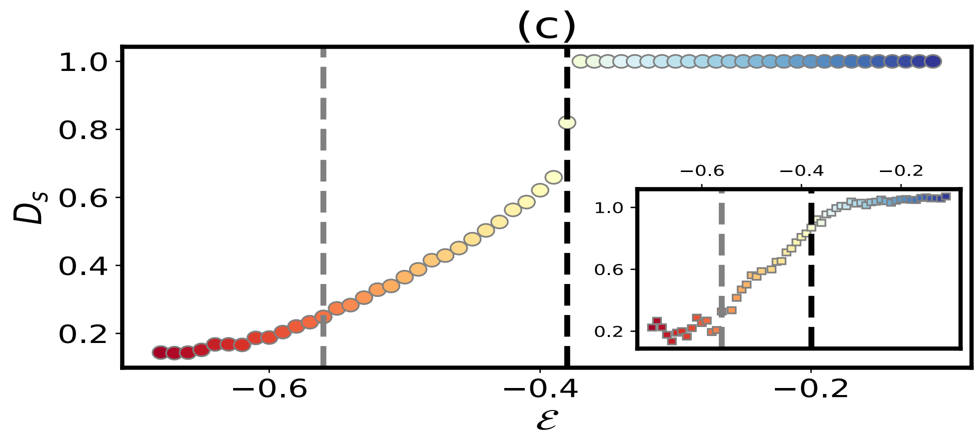
← Single-particle Green's function

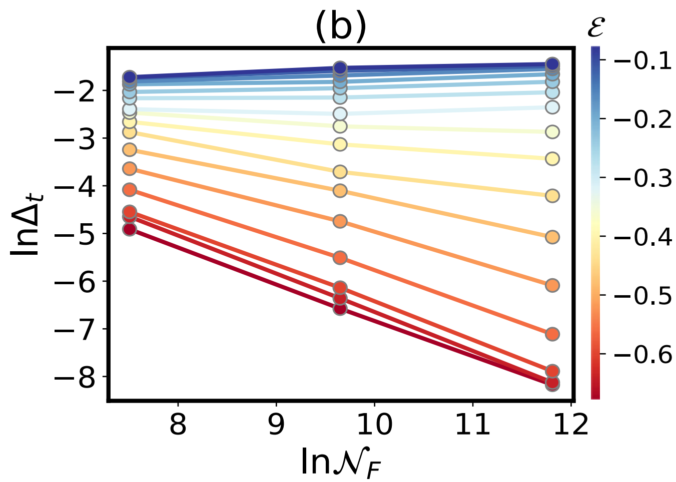
$$G^R_{i,n}(t) = -i\langle\Psi_n|\{c_i(t), c_i^\dagger(0)\}|\Psi_n\rangle\theta(t)$$

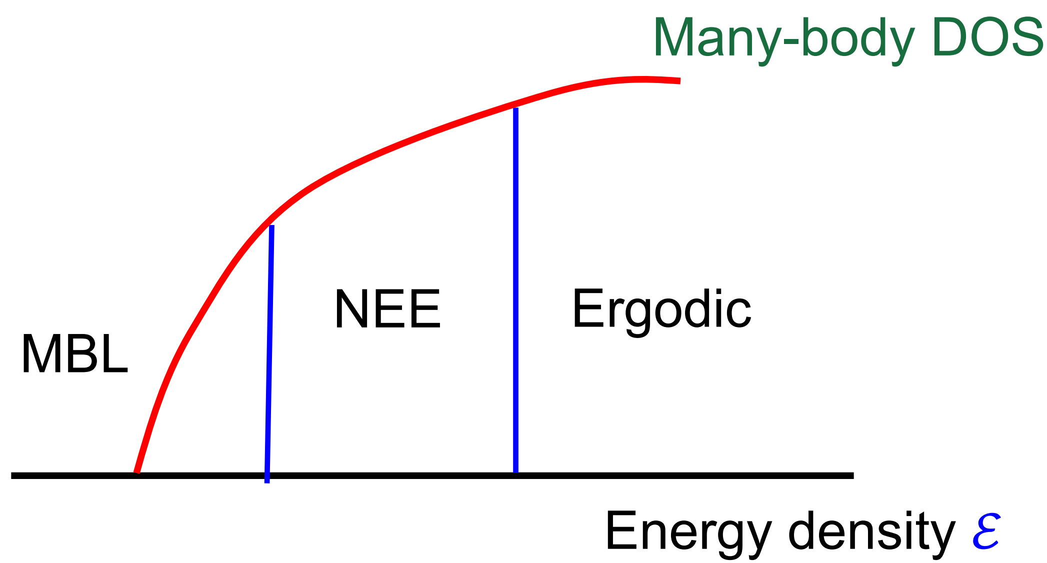
Fock-space transitions as function of energy density



Fixed quasiperiodic strength h







MBL and NEE states are both multifractal, cannot be distinguished via Δ_t

Δ_t is a probabilistic order parameter

Finite-size scaling theory, Asymmetric scaling ansatz (Garcia-Mata et al, PRL (2017))

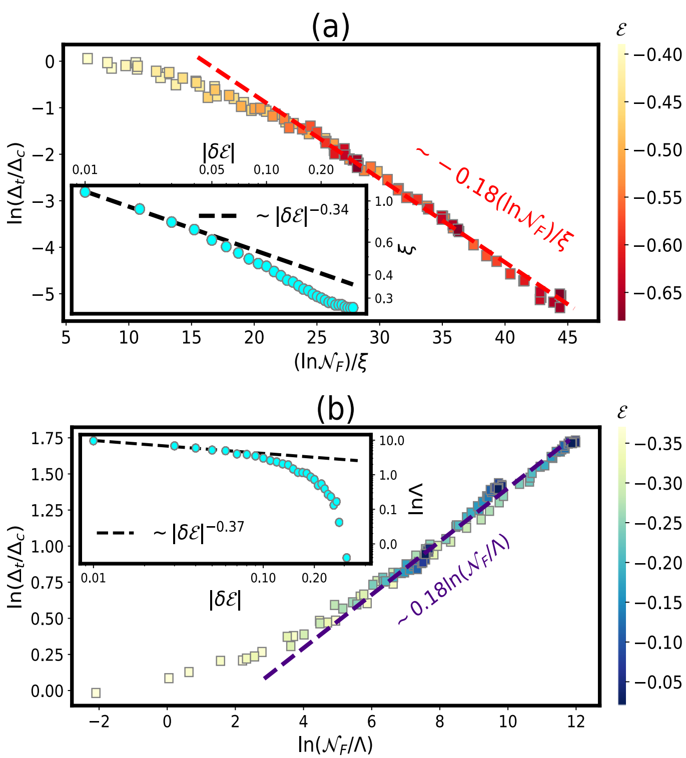
$$\ln\left(\frac{\Delta_t}{\Delta_c}\right) = \mathcal{F}_{vol}\left(\frac{N_F}{\Lambda}\right) \quad \mathcal{E} < \mathcal{E}_c \qquad \Rightarrow \mathcal{F}_{vol}\left(\frac{e^L}{\Lambda}\right) \quad \text{“Volumic scaling”}$$

$$= \mathcal{F}_{lin}\left(\frac{\ln N_F}{\xi}\right) \quad \mathcal{E} < \mathcal{E}_c \qquad \Rightarrow \mathcal{F}_{lin}\left(\frac{L}{\xi}\right) \quad \text{“Linear scaling”}$$

$$\Delta_c = \Delta_t(W_c) \sim N_F^{-(1-D_c)}$$

*For real-space $\mathcal{F}_{vol}\left(\frac{L^d}{\Lambda}\right) = \mathcal{F}_{lin}\left(\frac{L}{\Lambda^{1/d}}\right)$,
FS volume is exponential in L

Finite-size scaling collapse



$$\mathcal{E}_c \sim -0.38$$

$$\Lambda \sim \exp\left[\frac{b}{(W_c - W)^{0.4}}\right] \quad \text{“KT like”}$$

$$\xi \sim (W - W_c)^{-\nu} \quad \nu \simeq 0.34 \leq 1$$

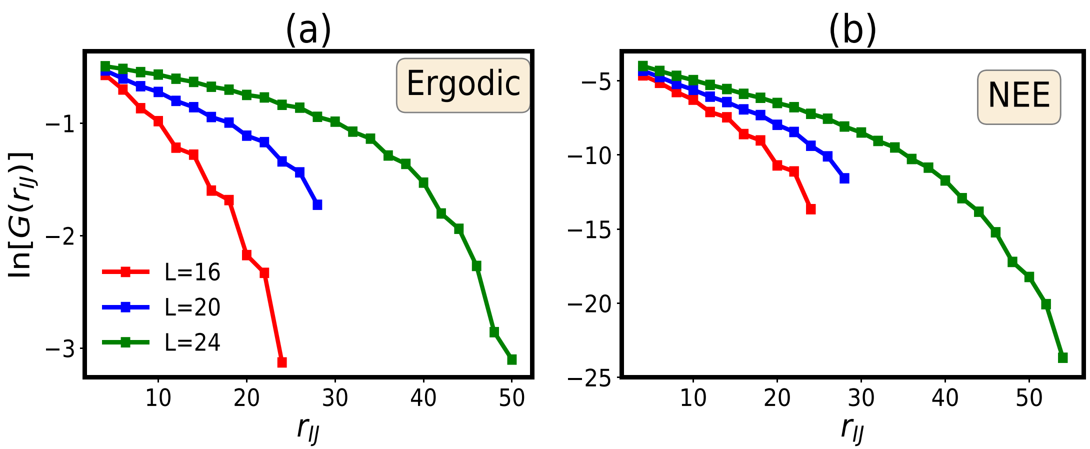
$$\Delta_t(W_c) \sim \Lambda^{-(1-D_c)} \quad N_F \to \infty$$

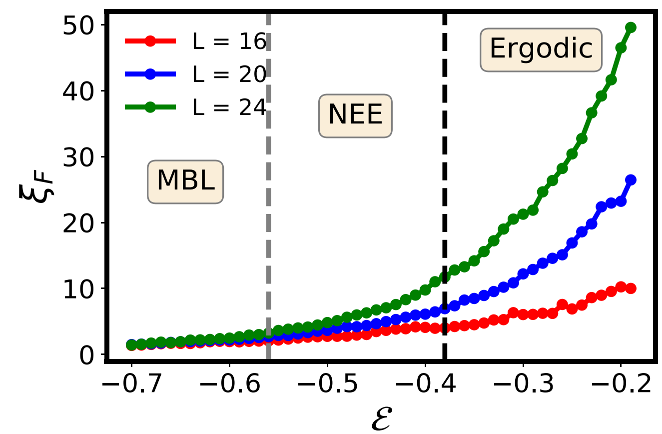
Asymptotic scaling functions for $x \gg 1$
($x = N_F/\Lambda$ or $\ln N_F/\xi$)

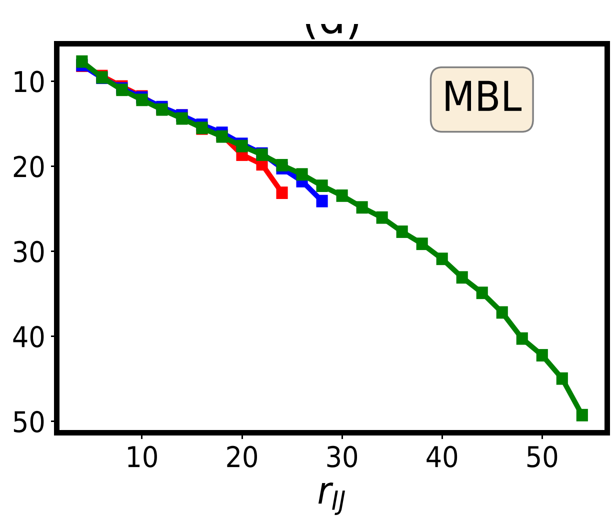
$$\mathcal{F}_{vol}(x) \sim (1 - D_c)\ln x$$

$$\mathcal{F}_{lin}(x) \sim -(1 - D_c)x$$

MBL and NEE states can be distinguished in terms of Fock-space localization length



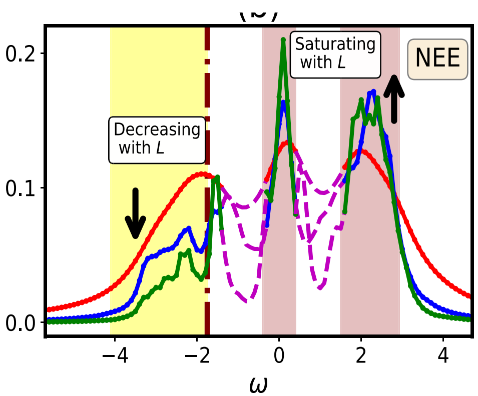


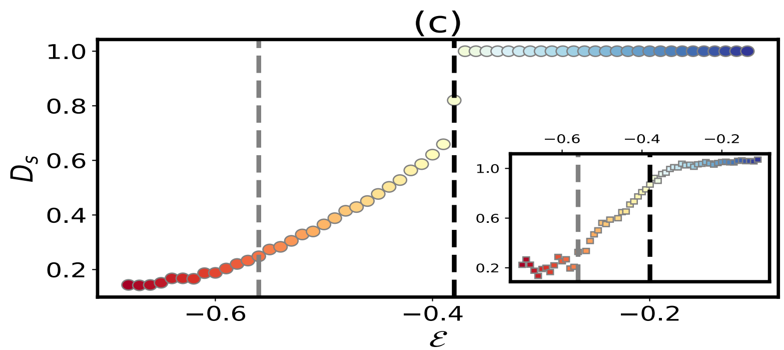


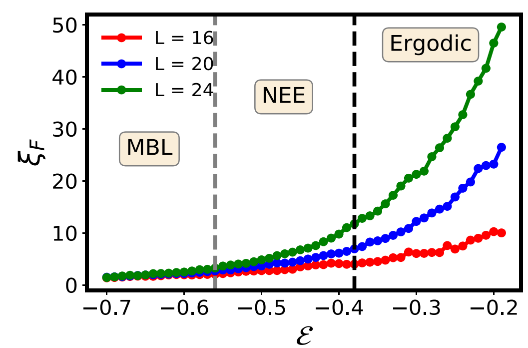
$\xi_F(L)$ depends on L in the NEE and ergodic phases but, independent of L in the MBL phase

Conclusions

Phases	Diagnostics						
	Entanglement entropy	Subsystem number fluctuations	Level spacing statistics	Single-particle (SP) excitations	Inverse participation ratio (IPR)	Typical local FS self energy Δ_t	FS localization length ξ_F
MBL	Area law	Finite for $L \to \infty$	Poisson	All SP excitations localized	$IPR \sim \mathcal{N}_F^{-D_2}$ $0 < D_2 < 1$ (Multifractal)	$\Delta_t \sim \mathcal{N}_F^{-(1-D_s)}$ $0 < D_s < 1$ (Multifractal)	ξ_F independent of L
NEE	Volume law	Finite for $L \to \infty$	Intermediate between Poisson and GOE	Both localized and delocalized SP excitations separated by an SP mobility edge	$IPR \sim \mathcal{N}_F^{-D_2}$ $0 < D_2 < 1$ (Multifractal)	$\Delta_t \sim \mathcal{N}_F^{-(1-D_s)}$ $0 < D_s < 1$ (Multifractal)	ξ_F increases with L
Ergodic	Volume law	Vanishes for $L \to \infty$	GOE	All SP excitations delocalized	$IPR \sim \mathcal{N}_F^{-1}$ (Extended)	$\Delta_t \sim \mathcal{O}(1)$ (Extended)	ξ_F increases with L



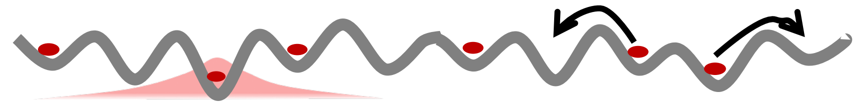




Thank You!

MBL: Real-space picture

$$H = \sum_{\alpha} \epsilon_\alpha c_\alpha^\dagger c_\alpha + \sum_{\alpha\beta\gamma\delta} V_{\alpha\beta\gamma\delta} c_\alpha^\dagger c_\beta^\dagger c_\gamma c_\delta$$



Huse & Oganesyan (2013)
Serbyn et al. (2013)

MBL fixed point → Emergent integrability
→ Quasi local integrals of motion
(LIOMs or "l-bits")

$$\tilde{n}_\alpha = \tilde{c}_\alpha^\dagger \tilde{c}_\alpha$$

$$\tilde{c}_\alpha^\dagger = \sum_i A_\alpha(i) c_i^\dagger + \cdots$$

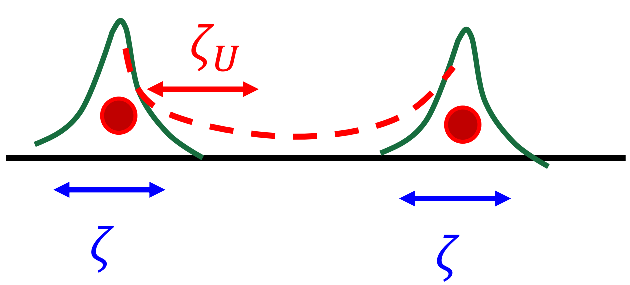
$$H_l = \sum_\alpha \epsilon_\alpha \tilde{n}_\alpha + \sum_{\alpha\beta} U_{\alpha\beta}\, \tilde{n}_\alpha \tilde{n}_\beta + \cdots$$

$$A_\alpha(i) \sim \exp[-\frac{|i - i_\alpha|}{\zeta}]$$

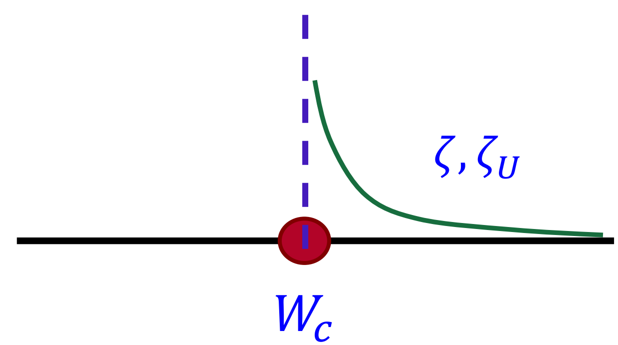
$$U_{\alpha\beta} \sim \exp\left[-\frac{|i_\alpha - i_\beta|}{\zeta_U}\right]$$

Perturbative construction

$$\tilde{c}_\alpha^\dagger \simeq c_\alpha^\dagger + \sum_{\beta\gamma\delta} \frac{V_{\delta\gamma\beta\alpha}}{\epsilon_\alpha + \epsilon_\beta - \epsilon_\gamma - \epsilon_\delta} c_\delta^\dagger c_\gamma^\dagger c_\beta + \cdots$$



Do $\zeta, \zeta_U \to \infty$
for $W \to W_c^+$?



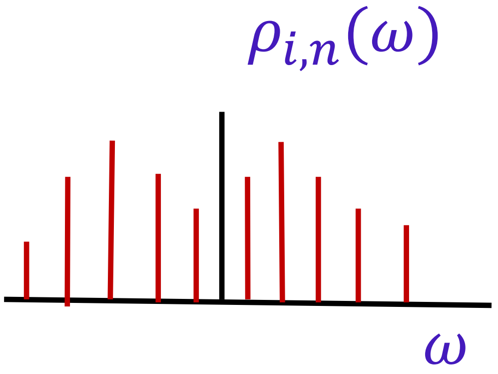
Spectral signatures of MBL

- Local spectral function of an eigenstate

$\rho_{i,n}(\omega)$

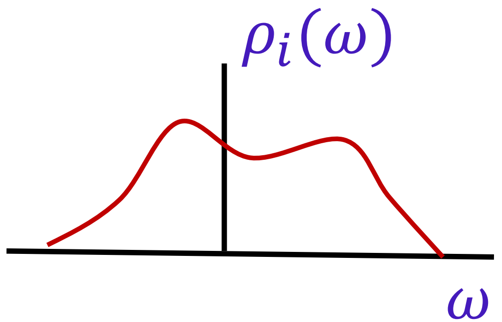
← Single-particle Green's function

$G^R_{i,n}(t) = -i\langle\Psi_n|\{c_i(t), c_i^\dagger(0)\}|\Psi_n\rangle\theta(t)$



- Thermal spectral function

$$\rho_i^{th}(\omega) = \frac{1}{Z}\sum_n e^{-\beta E_n}\,\rho_{i,n}(\omega)$$



* Generically, $\rho_i^{th}(\omega)$ does not contain information about localization at $T \neq 0$

Infinite temperature ($\beta \to 0$), $\rho_i^{th}(\omega) = (1/D)\sum_n \rho_{i,n}(\omega)$

Hilbert space dimension D

- Delocalized state, $\rho_i^{th}(\omega) = \rho_{i,n}(\omega)$ for $T \to \langle E\rangle = E_n$

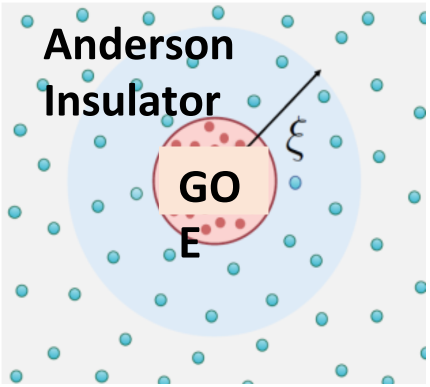
← ETH

Controversies and Caveats

- Till ~2020, the big question was about the nature of MBL transition

ED $\Rightarrow \nu \leq 1 < 2/d$ Phenomenological RGs $\nu \simeq 2.5 - 3 > 2/d$

- MBL unstable for $d > 1$ due to rare regions of weak disorder (thermal bubble)
De Roeck and Huveneers, PRB (2017)



Indications of chaos in the MBL phase
Suntajs et al, PRE, PRB (2020)

Numerical evidences of delocalizatin in the MBL phase
Sels, Polkovnikov et al.

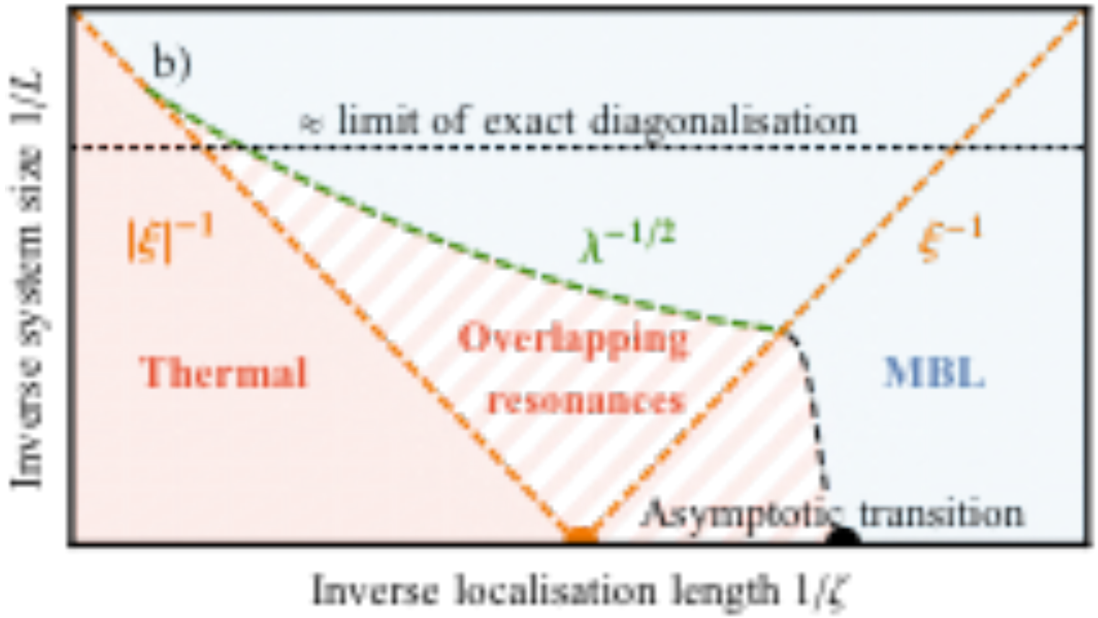
Long-range resonances in the MBL spectrum
Mornigstar, Huse et al.

MBL maybe unstable in the thermodynamic limit at any finite disorder even in 1d

vs.

MBL transition shifted much larger W_c

Finite-size MBL phenomenology



Crowley & Chandran, Sci. Post. (2022)