# Meeting of Random Walks & Consensus Dynamics on Random Directed Graphs

**LUCA AVENA** (DIMAI, Florence)

UNIVERSITÀ DEGLI STUDI FIRENZE
DIMAI
DIPARTIMENTO DI MATEMATICA E INFORMATICA "ULISSE DINI"

NETWORKS

**ICTS-NETWORKS workshop "Challenges in Networks", Bengaluru, January 30, 2024.**

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**joint work with Federico Capannoli, Rajat Hazra and Matteo Quattropani**

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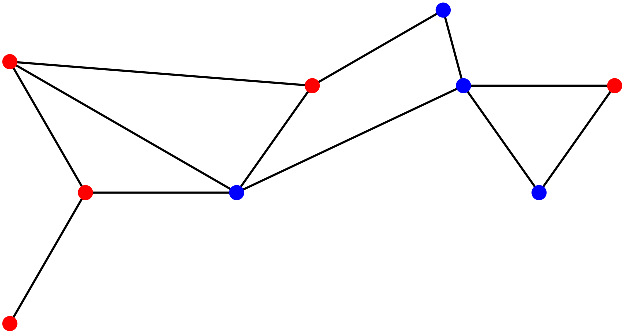

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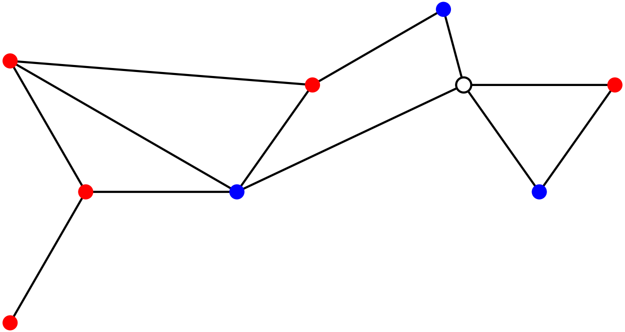

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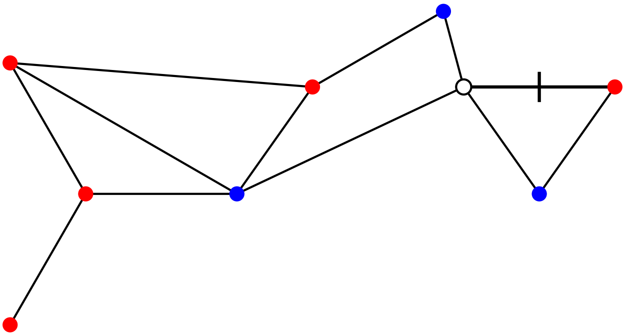

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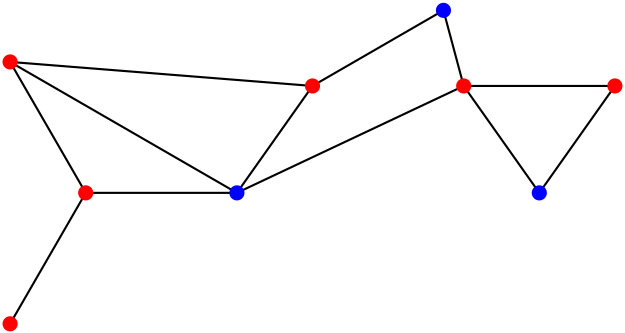

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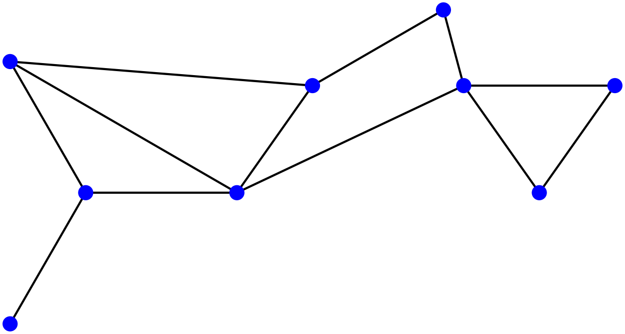

**Two absorbing states:** monochromatic configurations $\bar{\mathbf{1}}, \bar{\mathbf{0}} \in \{0,1\}^V$.

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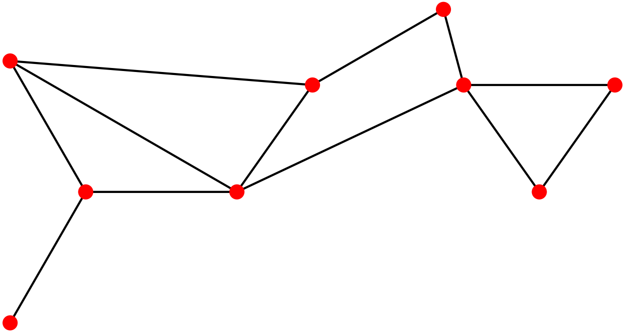

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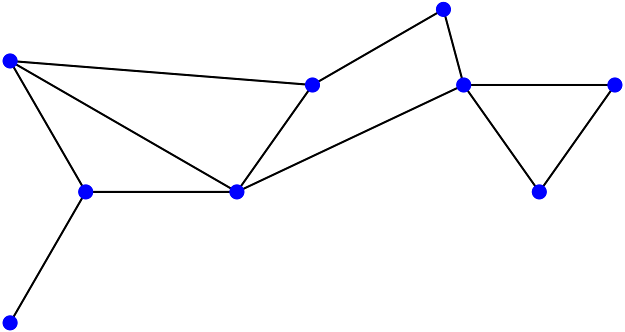

**How does the system reach consensus?**

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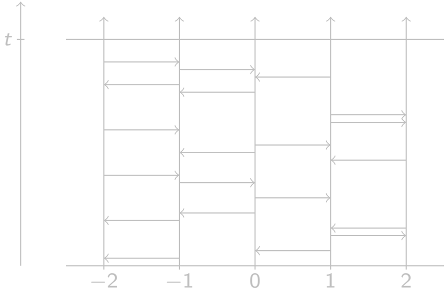

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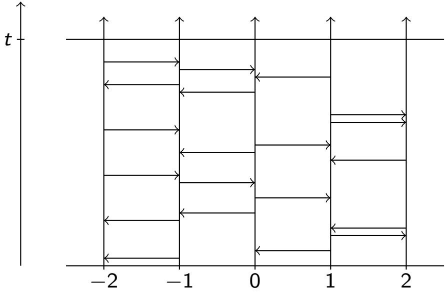

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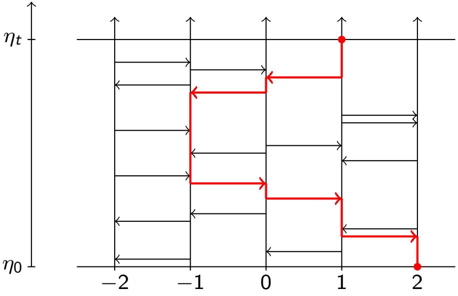


**Example:** The **red path** says that $\eta_t(1)$ is equal to $\eta_0(2)$.

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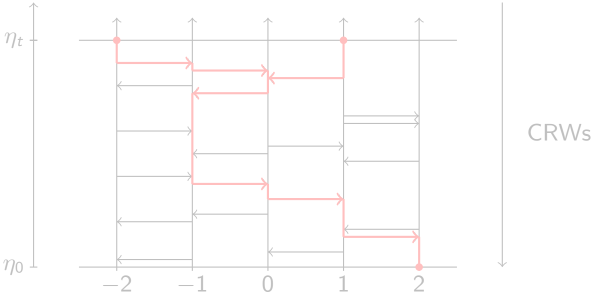

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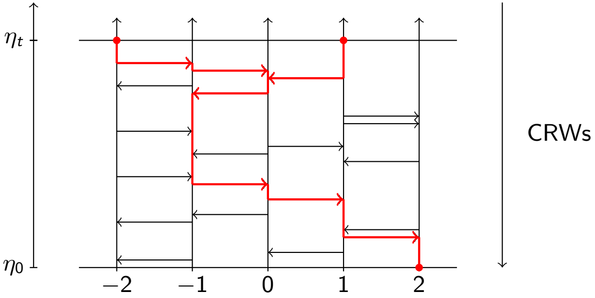

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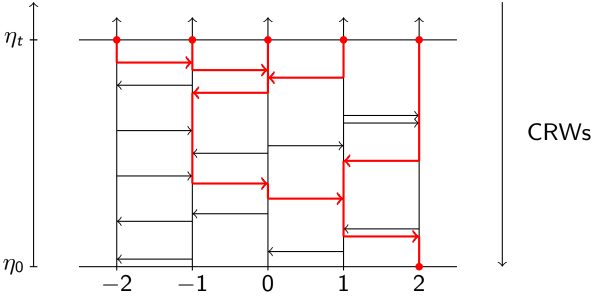


***n*-Coalescence & Consensus times:**

On $G$ finite, any two RWs meet in finite time. Thus, a.s., the *coalescence time*

$$\tau_{\text{coal}} := \inf\{t \geq 0 \colon n\text{-RWs coalesce}\} < \infty, \quad \& \quad \tau_{\text{cons}} \leq \tau_{\text{coal}} < \infty.$$

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$$\frac{\tau_{\text{coal}}}{\mathbf{E}[\tau_{\text{meet}}^{\pi\otimes\pi}]} \Rightarrow Z := \bigoplus_{k\geq 2} \exp\binom{k}{2}, \qquad \frac{\mathbf{E}\left[\tau_{\text{coal}}\right]}{\mathbf{E}\left[\tau_{\text{meet}}^{\pi\otimes\pi}\right]} \longrightarrow 2.$$

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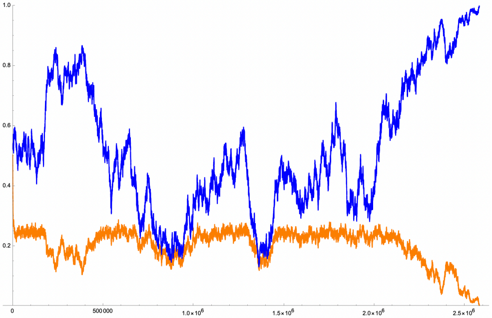
# Wright-Fisher approx.: Voter on $d$-regular Random Graphs

**Evolution until Consensus (“at time-scale $\mathbf{E}[\tau_{\text{meet}}^{\pi\otimes\pi}]$”)**




*Simulation with: time-steps* $\approx 10^6$ , *graph size* $= 10^3, u = 0.5, d = 3.$

- **Blue curve:** density of Blue opinions $\sum_{x\in[n]} \eta_t(x)/n$, with starting $\eta_0$ sampled from i.i.d. Bernoulli's of density $u$.
- **Orange curve:** density of discordant edges.

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- **(Aldous, Durrett, etc...)** For graphs with local weak limit being a supercritical Galton-Watson to be expected order $n$ meeting with pre-constant given by mean observable of G-W limit.
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# Meetings of Random Walks

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# on Sparse

# Random Digraphs

# (Directed Configuration Model)

# (Sparse) Directed Configuration Model - DCM($\mathbf{d}^{\pm}$)

Consider a fixed bi-degree sequence $\mathbf{d}^{\pm} = (d_x^+, d_x^-)_{x \in [n]}$ such that

- $m := \sum_{x \in [n]} d_x^+ = \sum_{x \in [n]} d_x^-$, **(graphical)**
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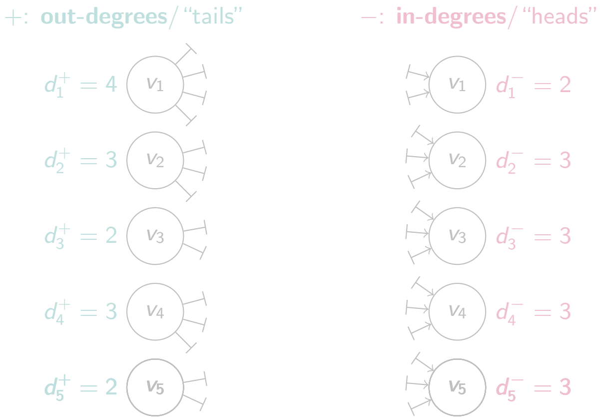

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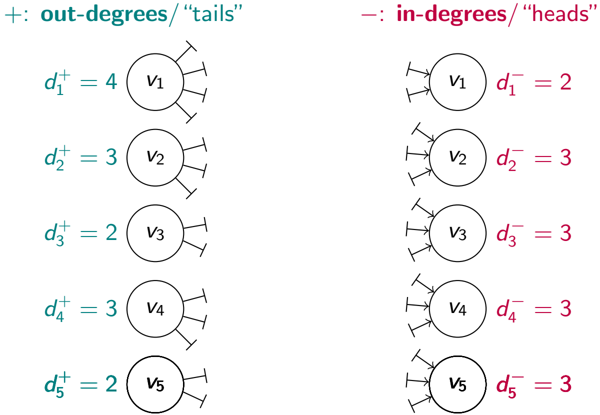

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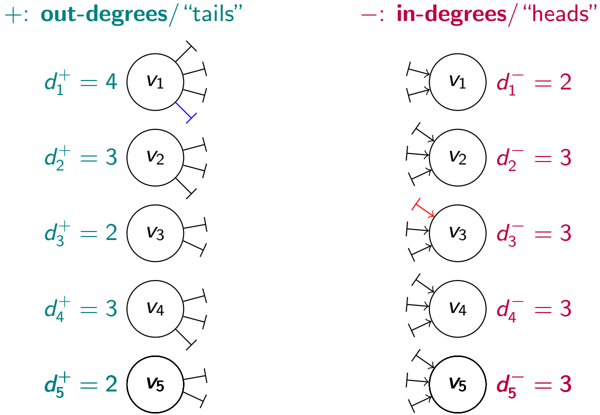

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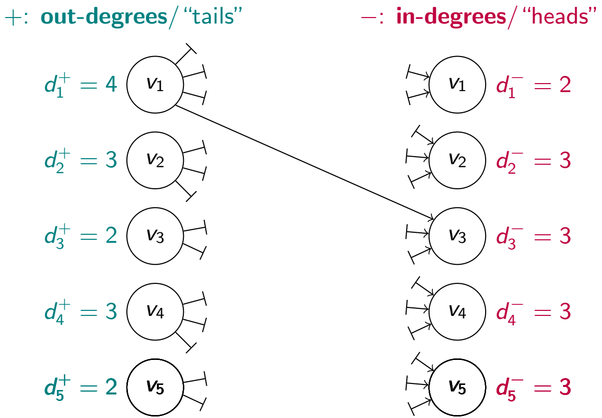

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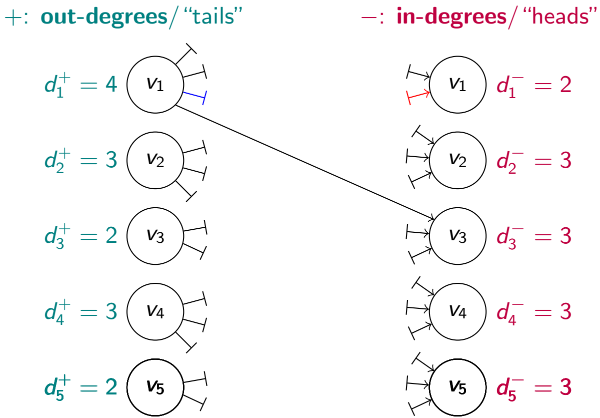

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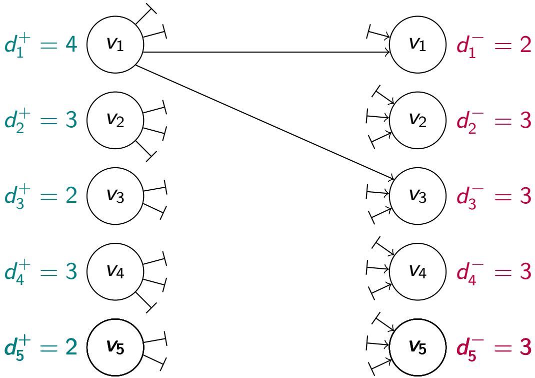

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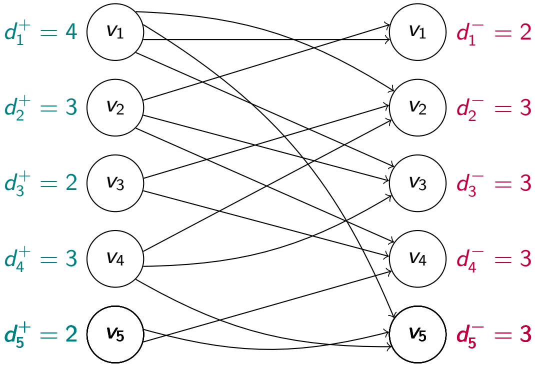

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# From Meeting to Hitting times: collapsed graph $\tilde{G}$

Take two copies of realized graph $G := ([n], E)$, to generate the product graph

$$G^{\otimes 2} := G \times G = (V^{\otimes 2}, E^{\otimes 2})$$

with $V^{\otimes 2} = \{(x, y) : x, y \in [n]\}$ and $E^{\otimes 2}$ such that

$$(x, y) \to (w, z) \iff \begin{cases} x \to w \text{ and } y = z, \text{ or} \\ y \to z \text{ and } x = w. \end{cases}$$

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such that $\tilde{V} = \{(x, y) \in V^{\otimes 2} : x \neq y\} \cup \Delta$ and all vertices in $\Delta$ retain the *in*- and *out*-stubs with their multiplicity.

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# First-Visit-Time-Lemma: "hitting times from stationarity"

**"Given a chain $\tilde{X}$ and a target state $\Delta$, if the chain mixes fast compared to the stationary mass of $\Delta$, then the hitting time of $\Delta$ is well approximated by a geometric whose parameter depends only on $\tilde{\pi}(\Delta)$ and on the local geometry around $\Delta$."**

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Lemma (Cooper, Frieze (2005) Manzo, Quattropani, Scoppola (2021))

*Consider a sequence of irreducible Markov chains on $N$ states with transition matrices $\tilde{P}_N$ and invariant measures $\tilde{\pi}_N$. Assume that*

1. *There exists some sequence of times $T = T(N)$ such that*
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   - $\max_{x\in[N]} T\,\tilde{\pi}_N(x) = o(1)$.
2. $\min_{x\in[N]} N^2\,\tilde{\pi}_N(x) \to \infty$.

*Then, for any fixed target $\Delta \in [N]$, its first hitting time $H_\Delta$ satisfies:*

$$\sup_{t\geq 0}\left|\frac{\tilde{\mathrm{P}}_{\tilde{\pi}_N}(H_\Delta > t)}{(1-\lambda)^t} - 1\right| \to 0, \qquad \frac{\lambda}{\tilde{\pi}_N(\Delta)/R_\Delta^T} \to 1,$$

*with*

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We want to compute local time at $\Delta$ up to $T = \log^4 n \, (\geq \tilde{t}_{\mathrm{mix}})$:

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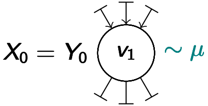
# Idea: Tree Construction for 1st Return in Coupled Rooted Forest

Start from the empty matching of the graph (Eulerian case)

Ex. $\mathbb{P}^{\mathrm{ann}}_{\mu}(H_{\Delta} = 4)$



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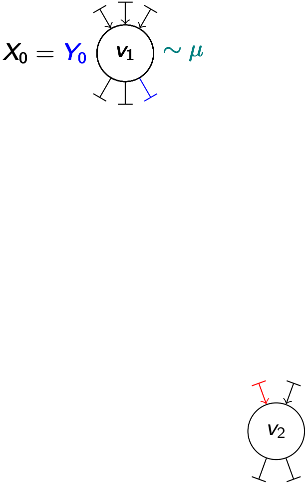

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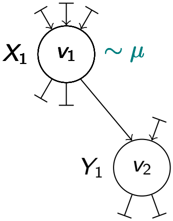

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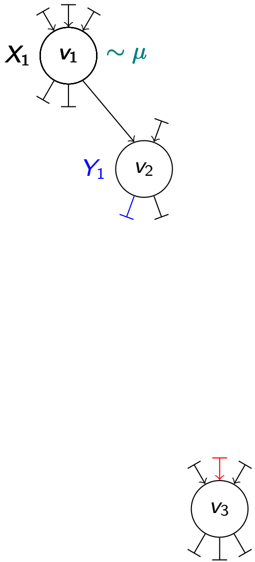

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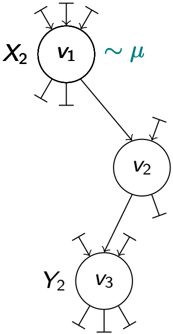

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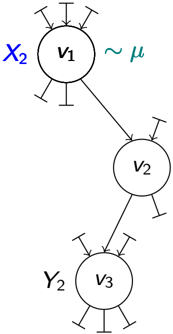

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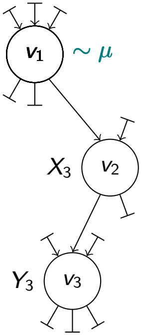

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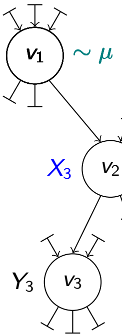

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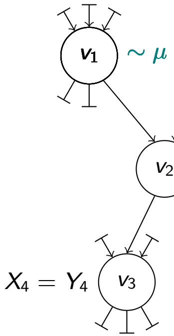
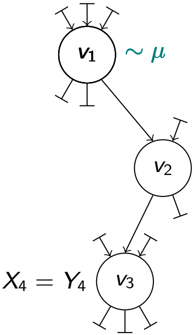


# Idea: Tree Construction for 1st Return in Coupled Rooted Forest

Denoting by $D_i$ the out-*offspring distribution* of $v_i$, $D_1 \sim \mu$ and $D_i \sim \mu^{+}_{\text{biased}}, i \neq 1$.




$$\mathbb{P}^{\text{ann}}_{\mu}(H_\Delta = 2t) = 2^{-2t+1}\frac{1}{t}\binom{2t-2}{t-1} f(\{D_i\}_{i \le t-1}).$$

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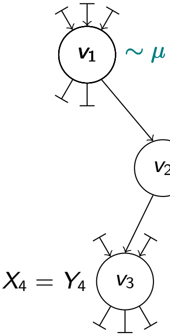


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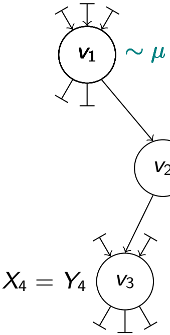


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# Thanks !