Looking Ahead

Number Fields

Other Number Fields include

$\mathbb{Z}(e^{2\pi i/3})$ $\mathbb{Z}(\sqrt{2})$

Should be a 'sector'

result in any such Field

e.g. a sector in $\mathbb{Z}[\sqrt{2}]$

could be

$$\left\{ a + b\sqrt{2} \;:\; 2 \le a/b \le 2 + \tfrac{1}{100} \right\}$$

[Submitted on 26 Jun 2012 (v1), last revised 1 Jul 2013 (this version, v2)]

A Density Version of the Vinogradov Three Primes Theorem

Xuancheng Shao

We prove that if A is a subset of the primes, and the lower density of A in the primes is larger than 5/8, then all sufficiently large odd positive integers can be written as the sum of three primes in A. The constant 5/8 in this statement is the best possible.

In $\mathbb{Z}[i]$, there should a universal density for all sectors.

Exceptional zeros, sieve parity, Goldbach

John B. Friedlander and Henryk Iwaniec

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Abstract

We survey connections between the possible existence of exceptional real zeros of Dirichlet L-functions and the sieve parity barrier and then show how recent work tying them to the Goldbach problem can be viewed in a considerably generalized framework.

Connections between Binary Goldbach and exceptional zeros is unexplored (in sectors)

Mathematics > Number Theory

[Submitted on 31 Dec 2020 (v1), last revised 4 Apr 2022 (this version, v2)]

Constellations in prime elements of number fields

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Number field analog to Szemeredi

Given any number field, we prove that there exist arbitrarily shaped constellations consisting of pairwise non-associate prime elements of the ring of integers. This result extends the celebrated Green-Tao theorem on arithmetic progressions of rational primes and Tao's theorem on constellations of Gaussian primes. Furthermore, we prove a constellation theorem on prime representations of binary quadratic forms with integer coefficients. More precisely, for a non-degenerate primitive binary quadratic form F which is not negative definite, there exist arbitrarily shaped constellations consisting of pairs of integers (x, y) for which $F(x, y)$ is a rational prime. The latter theorem is obtained by extending the framework from the ring of integers to the pair of an order and its invertible fractional ideal.

Comments: Minor revision (v2), explanations brushed up, 149 pages, 4 figures
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The polynomial Szemeredi Thms are unexplored.

Even Sarkozy!

Sarkozy: $A \subset \{1, \dots, N\}$ w/ no two differing by square integer satisfy $|A| = o(N)$

Easy to prove from major arcs.

Task to get a 'good' estimate for a broad class of number fields

Except for Constellations, multilinear is not known in Numberfields, e.g. Peluse Theory

Very Vague Question:

Weyl For a.e. x and polynomial p,

$\{p(n)x\}$ is uniformly distributed mod 1.

What is Number field variant?

Improving or Sparse Bounds

- Triangle operator of Anderson Kumchev Palsson ?

- Many results are known for averages (Lacey, Han, Yang, Hughes, ...)

- For discrete singular integrals

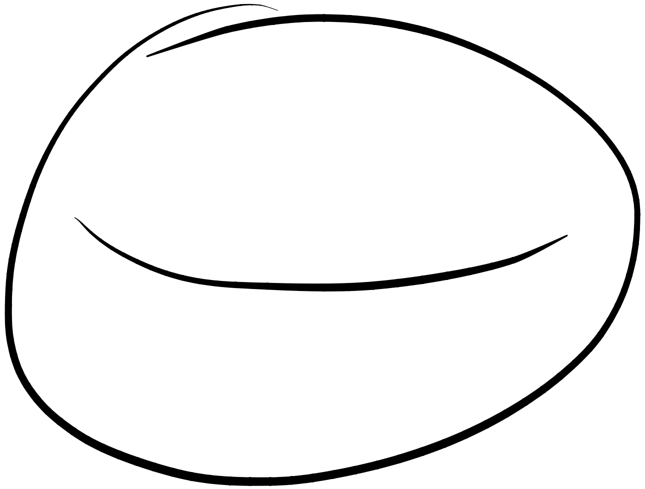
$$Hf(x) = \sum_{n \neq 0} \frac{f(x-n^3)}{n}$$

there is an ε-improvement in sparse bound. (Cladek - Henriot - Krause - Laba - Pramanik? Culiuc - Kesler - Lacey)

Can you prove $(\frac{3}{2}-\varepsilon, \frac{3}{2}-\varepsilon)$ for

$$\sum_{n \neq 0} \frac{f(x - \operatorname{sgn}(n)\, n^2)}{n} \; ?$$

- Can you establish good results for a sparse bound for Oscillation or variation of discrete averages?



Spheres

My advice: Attempt something new, such as recent work of Seeger et al.

Also Anderson Kumchev Palsson in sphere related

Question (Ben Krause)

Prove non-trivial estimates for the bilinear average

$e(\lambda(x-y))$ $e(\lambda(x+y))$

$$\int_{S^{d-1}} f(x-y)\, g(x+y)\, d\sigma$$

(It has a modulation invariance)

Ben Krause has proved

linear

$$\Big\| \sup_{\theta} \Big| \sum_{n \neq 0} e^{i\theta n^2} \frac{f(x-n)}{n} \Big| \Big\|_{\ell^2} \lesssim \|f\|_{\ell^2}$$

Quadratic

Could it be extended to

$$\Big\| \sup_{\theta} \Big| \sum_{n \neq 0} e^{i\theta n^2} \frac{f(x-n^3)}{n} \Big| \Big\|_{\ell^2} \lesssim \|f\|_{\ell^2}$$

two distinct powers

This inequality is equivalent to Carleson's Theorem on Fourier Series

$$\Big\| \sup_{\theta} \Big| \sum_{n \neq 0} e^{in\theta} \frac{f(x-n)}{n} \Big| \Big\|_{\ell^2} \lesssim \|f\|_{\ell^2}$$

is there an 'arithmetic' version? e.g.

$$\Big\| \sup_{\theta} \Big| \sum_{n \neq 0} e^{in^3\theta} \frac{f(x-n^3)}{n} \Big| \Big\|_{\ell^2} \lesssim \|f\|_{\ell^2}$$

Previous question is very hard. Can you establish

$$\Big\| \sup_{\theta \in \Theta} \Big| \sum_{n \neq 0} e^{in^2\theta} f(x-n^2) \Big| \Big\|_{\ell^2} \lesssim \|f\|_{\ell^2}$$

for non-trivial Θ ?

Random Averages.

Let X_n be indep. Bernoulli r.v.

with $\mathbb{P}(X_n = 1) = 1/n^\alpha \quad 0<\alpha<1$

$$A_n f(x) = \frac{1}{N^{1-\alpha}} \sum_{n=1}^{N} X_n f(x-n)$$

(Bourgain early arithmetic paper)

shows $Mf = \sup_N A_N : \ell^p \to \ell^p \quad 1 < p < +\infty.$

Q (Krause) For $\alpha = 1/2$ prove or disprove

$M : \ell^1 \to \ell^{1,\infty}$. a.s.

(For $\alpha > 1/2$, this is true)

(The probability event is a tail event, hence occurs w/ prob 0 or 1)

With same notation, define random

Y_K by $\sum_{1}^{Y_K} X_n = K$

Set $$Hf(x) = \sum_{1}^{\infty} \frac{f(x-Y_k) - f(x+Y_k)}{k}$$

For $\boxed{\alpha = 1/2}$ does $H : \ell^1 \to \ell^{1,\infty}$ a.s.?

Same Notation $\quad \alpha = 1/2$

(Bourgain, if I am remembering it correctly)

$$B(f,g) = \sum_k \frac{f(x-Y_k)\,g(x+Y_k)}{k}$$

Prove $\ell^2 \times \ell^2 \to \ell^1$. Same for bilinear

maximal $f^{\underline{n}}$.

Q: What is the threshold density of set of random differences $D \subset \{1, \dots, N\}$ so that ALL $A \subset \{1, \dots, N\}$ of density $\geq \frac{1}{100}$ (say) contain 3 points

$$x, \quad x+d, \quad x+2d \in A, \quad d \in D$$

density of $D \geq N^{-1/2}$ is relatively easy using a variant of the random bilinear averages. This paper breaks this square root barrier

ON THE THRESHOLD FOR SZEMERÉDI'S THEOREM WITH RANDOM DIFFERENCES

JOP BRIËT AND DAVI CASTRO-SILVA

Abstract. Using recent developments on the theory of locally decodable codes, we prove that the critical size for Szemerédi's theorem with random differences is bounded from above by $N^{1-\frac{2}{k}+o(1)}$ for length-k progressions. This improves the previous best bounds of $N^{1-\frac{1}{\lceil k/2 \rceil}+o(1)}$ for all odd k.

1. Introduction

Szemerédi [18] proved that dense sets of integers contain arbitrarily long arithmetic

This question has many variants

Q (Sarkozy variant)

What is the threshold density for random $D \subset \{1, \dots, \sqrt{N}\}$ so that all $A \subset \{1, \dots, N\}$ of density $> \frac{1}{100}$ say contain

$$x, \quad x + d^2, \in A \quad d \in D$$

Q (non linear Roth variant)

Same question for

$x, \; x+y, \; x+y^2 \in A \;,\; y \in D$

Critical cases for Peluse Theorem

Prove: $\exists c>0 \quad \forall A \subset \{1, \ldots N\}$ of density $(\log\log N)^{-c}$ contain

- $x, \ x+y^2, \ x+2y^2$

- $x, \ x+y, \ x+2y, \ x+y^2$