Vertex operator algebras (VOAs) of 4d $\mathcal{N} \geq 2$ Superconformal Field Theories (SCFTs)

Lecture 1

1st lecture: Introducing the players

2nd lecture: map:

$$\{ 4d\ \underline{\mathcal{N}=2\ \text{SCFTs}} \} \longrightarrow \{ \text{VOAs} \}$$

3rd lecture: props. of map & consequences for 4d physics.

What is a theory?

Give a Lagrangian description in terms of a gauge theory $G = G_1 \times \cdots \times G_n$.

Gauge fields — Field strength $F_{\mu\nu}$ in adj of G

$\mathcal{N}=2$ SUSY — supercharges Q^{i}_{α} $i = 1, 2$
$\alpha = \pm$

$$\tilde{Q}_{\dot\alpha = \pm} \qquad i = 1, 2.$$

complex scalar	ϕ — $\Delta = 1$
2 Weyl Fermions	$\lambda^{i=1,2}_{\alpha}$ — $\Delta = 3/2$
$\Delta = 2$ —	$F_{\mu\nu}$

vector multiplet.

in adjoint of G

$\mathcal{N}=2$ SUSY requires this - rep. — more soon.

$\Rightarrow$ Free fields $\quad \Box\phi = 0, \; - - - -$

Add matter in some rep. of G — only one type of matter.

scalar $\rightarrow$ $q^{i=1,2}$	$\Delta = 1$
ψ_α	$\tilde{\psi}_{\dot\alpha}$

$\Delta = 3/2$

half-hypermultiplet

in some rep R of G

add CPT conjugate in rep $\bar{R}$ of G.

$\Box q^{i=1,2} = 0$, shortening cond $Q_\alpha^{(i} q^{j)} = 0$

Write Lagrangian: add vectors in adjoint of G

+ hypers in reps of G.

impose conformality $\Rightarrow$ impose β-function $=0$

- fixes what matter you can have - classify

$\mathcal{L}$ preserving both susy + conformal

— Bhardwaj Tachikawa '13

fixed up to the complexified gauge couplings

— parametrize the conformal manifold.

Operators in the theories - gauge invariant combinations of free fields, e.g.

$$M^{ij} = q^{(i a} \tilde{q}^{j)}_{a}$$

rep of G

$\langle M^{ij} M^{kl} \dots \rangle$ can try to compute corr. fns - observables in theory.

Start from one of these $\mathcal{L}$ theories

- increase the coupling - find a region where now the theory admits a <u>dual</u> description in terms of new building blocks that do not admit a conventional Lagrangian description.

Free vectors + hypers $\xrightarrow{\text{increase coupling}}$ dual description

<u>isolated</u> <u>theory</u> that has some flavor symmetry e.g. Gaiotto's T_N

gauge weakly this flavor symm.

This motivates us to take an abstract approach:

describe theory

$\{\mathcal{O}\}$ ← set of all ops

e.g. $H^{(i)}$

$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle$ — corr fns $=$ observables.

Operators must lie in representations of any symm. algebra.

Representation theory of $\mathcal{N}=2$ SCA

quick summary of conformal algebra & its reps.

Conf algebra is $SO(4,2)$

generators are $M_{\mu\nu}$, D, P_μ, K_μ

Conformal algebra

$\mu, \nu, \ldots = 0, \ldots, d-1$

metric on $\mathbb{R}^{3,1}$

$$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho}$$

↳ Lorentz algebra $so(d-1,1)$

$$\left.\begin{aligned} [M_{\mu\nu}, P_\rho] &= \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu \\ [M_{\mu\nu}, K_\rho] &= \eta_{\nu\rho} K_\mu - \eta_{\mu\rho} K_\nu \end{aligned}\right\} \quad P_\mu, K_\mu \text{ Lorentz vectors}$$

$$[K_\mu, P_\nu] = 2(\eta_{\mu\nu} D - M_{\mu\nu})$$

dilation op.

$$\left.\begin{aligned} [D, P_\mu] &= P_\mu \\ [D, K_\mu] &= -K_\mu \end{aligned}\right] \quad P_\mu, K_\mu \text{ act as lowering \& raising ops for } D$$

all other $[\,,\,] = 0$.

conf primary $\quad [D, \mathcal{O}(0)] = \Delta \, \mathcal{O}(0)$

$\uparrow$ scaling dim.

0

$\uparrow K_\mu$

rep of Lorentz $\rightarrow \mathcal{O}_{\mu_1, \dots, \mu_\ell}(0) \rightarrow$ lowest weight state.

$\downarrow P_\mu$

$\Delta + 1 \longrightarrow \partial_\mu \mathcal{O}(0)$

$\downarrow$

$- - - - \qquad - - - -$

Conformal multiplet

add susy: $N = 2$ superconf algebra

$SU(2,2|2)$ – Lie super algebra i.e. $\mathbb{Z}_2$ graded algebra $\quad g = g_0 \oplus g_1 \quad$ w/

g_0: bosonic, even

g_1: fermionic, odd

$[\ ,\ \} : g \times g \to g$ bilinear product.

$[a_i, b_j\} = -(-1)^{ij} [b_j, a_i\}$ $\quad a_i \in \underline{g}_i$ $\quad b_j \in g_j$

$\to [A,B] = AB - BA$ if A or B is in g_0 – bosonic

$\{A,B\} = AB + BA$ if A and B are in g_1 – fermionic

↳ either commutator or anti-commutator.

→ consistent w/ grading $[a_i, b_j\} \subset g_{i+j \bmod 2}$

$= g_0$ if $i = j$

g_1 if $i \neq j$

obeys graded Jacobi id

g_0 is a Lie algebra

$g_0 = \underbrace{SO(4,2)}_{\text{conf alg.}} \oplus \boxed{SU(2)_R \oplus U(1)_r}$ – R-symmetry commutes w/ conf alg.

conf alg. $\{P_\mu, K_\mu, M_{\mu\nu}, D\}$.

gen. by $\{R_+, R_-, R, r\}$

$\mathfrak{g}_1$ is a $\mathfrak{g}_0$- representation.

- superchargees - fermionic - nilpotent

$Q^{i=1,2}_{\alpha}$ ← $\mathcal{N}=2$ $\qquad \tilde{Q}_{i\dot{\alpha}}$ - Poincaré supercharges

$SU(2)_R$ rotates them & charged under $U(1)_r$

↳ spinors of Lorentz

S^{α}_{i} , $\tilde{S}^{\dot{\alpha} i}$ - conformal supercharges

$\mathcal{N}=2$ SCA: generated by

$$\{ Q, \tilde{Q}, S, \tilde{S}, \underbrace{R, R_+, R_-}_{SU(2)_R}, \underset{\uparrow \atop U(1)_r}{r}, K_\mu, P_\mu, M_{\mu\nu}, D \}$$

Representations of $\mathcal{N}=2$ SCA

$S, \tilde{S}$ lower dim by $1/2$

$Q, \tilde{Q}$ raise by $1/2$.

$\mathcal{O}(0)|0\rangle = |\mathcal{O}\rangle$ — state operator corr. in CFTs

local ops

states

$\{\tilde{S}^{i\dot\alpha}, S^{\alpha}_{j}\} = \frac{1}{2}\delta^{i}_{j} K^{\dot\alpha\alpha}$

0

$\uparrow S, \tilde{S}$

Superconf. primary

(also a conf primary)

$|\mathcal{O}^{R,r}_{\mu_1 \cdots \mu_\ell}\rangle$ — lowest weight state of Dimension Δ

$\downarrow Q, \tilde{Q}$

$|\mathcal{O}'\rangle$ — $\Delta + 1/2$

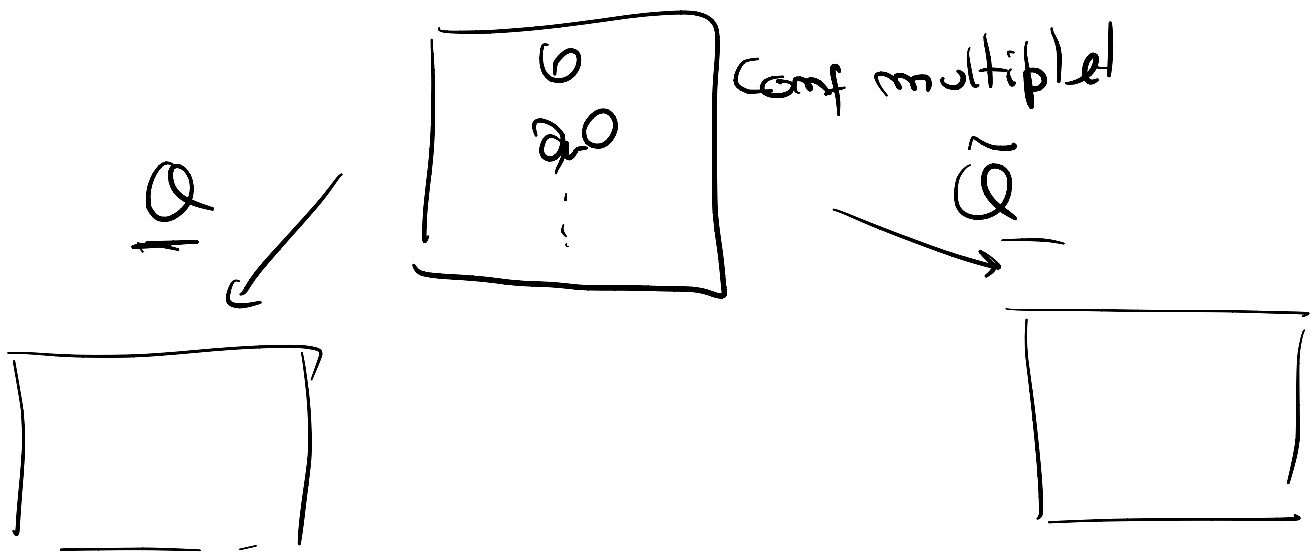
$\{Q^{i}_{\alpha}, \tilde{Q}_{j\dot\alpha}\} = \frac{1}{2}\delta^{i}_{j} P_{\alpha\dot\alpha}$

Some are conf. primaries and some are desc.

$$P_{\alpha\dot{\alpha}} = \sigma^{\mu}_{\alpha\dot{\alpha}} P_{\mu}.$$

Superconformal multiplet

$$= \sum^{\text{finite}} \text{Conf. multiplets}$$



Superconf multiplet is labeled by quantum #s of the superconf primary ($S|O\rangle = \tilde{S}|O\rangle = 0$)

$\{ \Delta,$ Lorentz rep $j, \bar{j}$ spins, $SU(2)_R$ R, $U(1)_r$ r $\}$

Unitary Lorentzian theories $\Rightarrow$ requires that all ops have positive norm.

superconf. prim $|O\rangle$

$\downarrow$ act w/ $Q, \tilde{Q}$

must have positive norm. $\Rightarrow$ superconf desc $\{ - - - - \}$

CFT : work in radial quantization.

hermitian op. is †

$$(Q^i_\alpha)^\dagger = S^\alpha_i \qquad (\tilde{Q}_{i\dot\alpha})^\dagger = \tilde{S}^{i\dot\alpha}$$

$$P_\mu^\dagger = K_\mu \qquad \ldots$$

$$(Q^i_\alpha|0\rangle)^\dagger (Q^i_\beta|0\rangle) \rightarrow$$ require positive semi-def matrix of inner prods.

Unitarity bounds: $\Delta \geq \Delta_B(j, \bar{j}, R, r)$

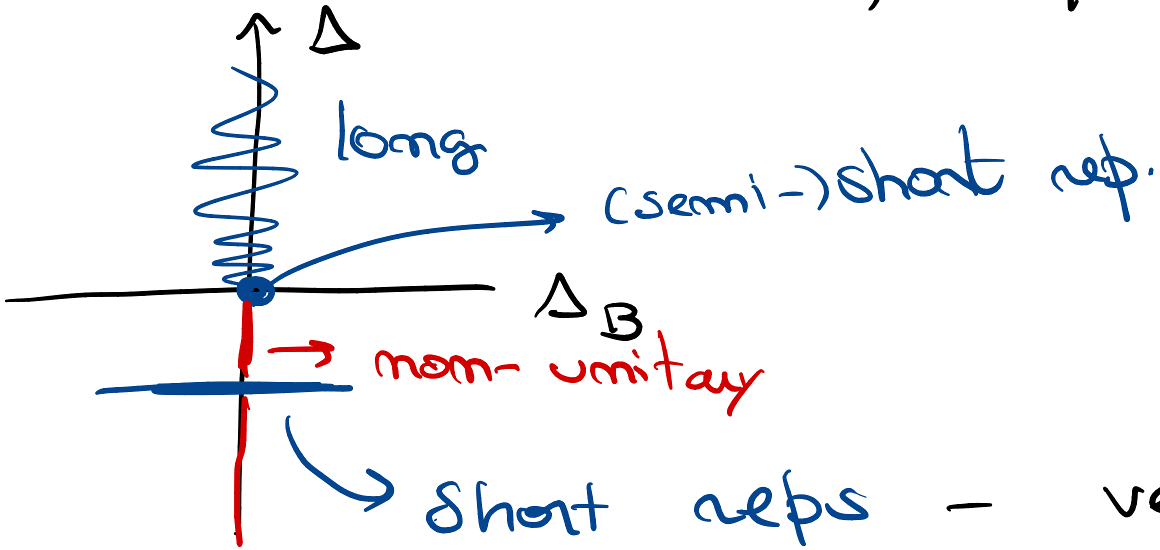
$>$ - long multiplet

$=$ - short multiplet - null state - removed to get irreducible rep.

q^i $\qquad \Box q^i = 0$ — null state

$\Updownarrow$

$\lambda \qquad \bar{\lambda}$ $\qquad P_\mu P^\mu q^i = 0$ $\qquad$ e.o.m.



— vector & hyper were examples of these

- Dolan, Osborn 02; Cordova, Dumitrescu, Intriligator

Eberhardt — review these.

— all unitary reps have been classified — given name — type of shortening cond — which is the null state.

What is a theory? List of ops organized in superconf reps.

super prim. — $M^{(ij)}$ — short multiplet $\Delta = 2$

$\underline{3}$ of $SU(2)_R$, scalar, $r=0$, $\left(\mathcal{B}, \bar{\mathcal{B}}, [0;0]_2^{(2;0)} \right.$ $\left. \hat{\mathcal{B}}_1 \right)$

$$Q^{(K} M^{ij)} = 0$$

$\boxed{\text{More to come}}$!

SCA $\quad \underline{SU(2)}_{\underline{R}} \oplus U(1)_r$

$$[R^i{}_j , Q^k{}_\alpha] = \delta^k_j Q^i_\alpha - \frac{1}{4}\delta^i_j Q^k_\alpha$$

$SU(2)_R \oplus U(1)_r$ gen.

$r = -4(R^i{}_i) \quad - U(1)_r$

$$SU(2)_R \begin{cases} R_+ = R^1{}_2 \\ R_- = R^2{}_1 \\ 2R = R^1{}_1 - R^2{}_2 \end{cases}$$

Q^k_α is $\underline{2}$ of $SU(2)_R$ $\qquad \tilde{Q}_{\dot\alpha\,\underline{k}}$

$\hookrightarrow$