An effective version of Ratner's equidistribution theorem for $\mathrm{SL}(3, \mathbb{R})$

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Ratner's theorem (1991)

For any $x \in X$, the closure of $Ux = \{u(r)x : r \in \mathbb{R}\}$ is a closed orbit Lx of some Lie subgroup $L \subset G$. Moreover, the orbit Ux is equidistributed in Lx in the following sense: For any $f \in C_c^\infty(X)$,

$$\lim_{T \to \infty} \frac{1}{T} \int_{[T]} f(u(r)x) \mathrm{d}r = \int_{Lx} f(y) \mathrm{d}\mu_L(y).$$

Here $[T] = [-T/2, T/2]$, and μ_L denotes the unique L-invariant probability measure on Lx.

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Open Problem

Can we give an explicit upper bound on

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Dream Theorem

There exist $C, \eta > 0$ and some Sobolev norm $\|\cdot\|_S$ such that for any $x \in X$ either

$$\left|\frac{1}{T}\int_{[T]} f(u(r)x)\mathrm{d}r - \int_{X} f(y)\mathrm{d}\mu_G(y)\right| \leq C\|f\|_S T^{-\eta},$$

or $u([T])x$ is close to some proper closed orbit Lx.

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- $$u(r) = \begin{bmatrix} 1 & & \\ & 1 & r \\ & & 1 \end{bmatrix}$$

Theorem 1(Y. 2022)

There exist $C, \eta, T_0 > 0$ and some Sobolev norm $\|\cdot\|_S$ such that for any $T > T_0$ and $x \in X$, either

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or $b(\log T)x$ or $b'(\log T)x$ is "far in the cusp".

Obstruction to Effective Equidistribution

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- $b(t) = a(-t)a_0(t)$, $b'(t) = a(-t)a_0(-t)$

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- Obstruction: $b(\log T)x \notin X_{T^{-\kappa}}$ or $b'(\log T)x \notin X_{T^{-\kappa}}$ where $\kappa = \frac{1}{3} - 0.001$

Expanding Translates of Unipotent Orbits

Theorem 2(Y. 2022)

There exist $C, \eta, t_0 > 0$ and some Sobolev norm $\|\cdot\|_S$ such that for any $t > t_0$ and any $x \in X$, either

$$\left| \int_{[1]} f(a(t)u(r)x)\mathrm{d}r - \int_X f(y)\mathrm{d}\mu_G(y) \right| \leq C\|f\|_S e^{-\eta t},$$

or $a_0(t)x \notin X_{e^{-\kappa t}}$ or $a_0(-t)x \notin X_{e^{-\kappa t}}$.

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Let $\mathbf{f} : [1] \to \mathbb{R}^2$ be a non-degenerate C^3 curve and $x \in X$. There exist $C, \eta, t_0 > 0$ and some Sobolev norm $\| \cdot \|_S$ such that for any $t > t_0$,

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The ineffective version is proved by Shah (2009).

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$$\max\{\langle n w_1 \rangle, \langle n w_2 \rangle\} \geq C' n^{-\omega}.$$

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Let (w_1, w_2) be 0.6-Diophantine. There exist $C, \eta, t_0 > 0$ and some Sobolev norm $\| \cdot \|_S$ such that for any $t > t_0$,

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- Lindenstrauss-Margulis-Mohammadi-Shah (in progress): logarithmic effective density in general case

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- Kim (2021+): $G = \mathrm{SL}(n,\mathbb{R}) \ltimes \mathbb{R}^n$, $\Gamma = \mathrm{SL}(n,\mathbb{Z}) \ltimes \mathbb{Z}^n$, U horospherical in semisimple part

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- Lindenstrauss-Mohammadi-Wang (2022): polynomial effective equidistribution for $G = \mathrm{SL}(2,\mathbb{C}), \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$

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- The critical part of the approach is quite different from previous works;
- The framework can be applied to general cases.

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Venkatesh's argument

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- If we can prove that the normalized measure μ_s on $a(s)u([1])x$ has large dimension in the following sense: for any ball $B_{s'}(x)$ of radius $e^{-s'}$,

$$\mu_s(B(x)) \leq e^{-(d-\theta)s'}$$

where d is the dimension of the whole space and θ is a small constant, then we have that $a(s')_*\mu_s$ is effectively equidistributed.

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- Suppose μ_i is (d_i, s_i)-good, we want to prove μ_{i+1} is (d_{i+1}, s_{i+1})-good, where $d_{i+1} = d_i + \epsilon_1$, $s_{i+1} = s_i/2$;

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- When d_i is close to 4, we can apply Venkatesh's argument to get effective equidistribution.

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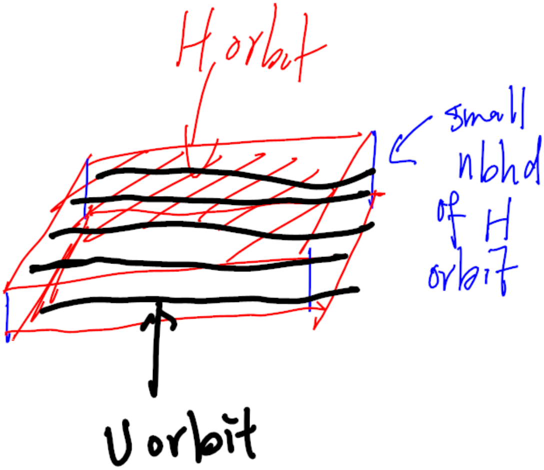

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Closing lemma(Einsiedler-Margulis-Venkatesh, 2009)

There exist $d_0, \xi > 0$ such that if μ_0 is not (d_0, s')-good, then the whole orbit $a(s)u([1])x$ is $e^{-\xi s'}$-close to a closed H-orbit

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Dimension Improvements

- Lindenstrauss-Mohammadi-Wang: Margulis function

$$f_i(x) := \sum \|\mathbf{w}\|^{-\alpha}$$

where $\mathbf{w}$ runs over all small vectors in the Lie algebra of G which is transversal to $Lie(H)$ such that $\exp(\mathbf{w})x \in \mathrm{supp}\mu_i$. Then prove that $\int_{[1]} f_{i+1}(a(\ell)u(r)x)\mathrm{d}r \le af_i(x) + b$.

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- Key: a Kakeya type model

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$$\mathcal{T}(y_1) \cap \mathcal{T}(y_2) \neq \emptyset.$$

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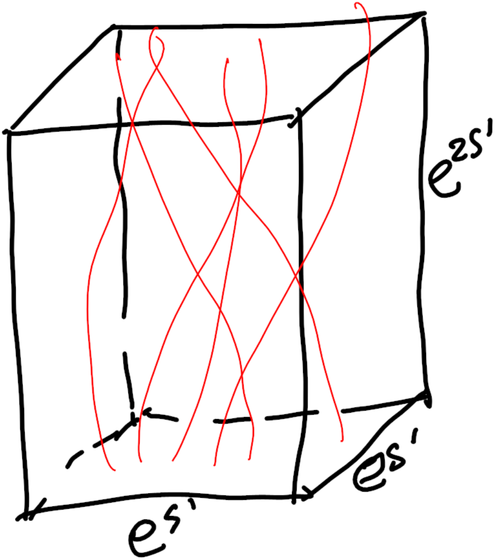

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- Those structures imply that the orbit is close to a closed periodic orbit.

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- Fractal Theory (Hochman): $H_n(\mu * \nu) \geq H_n(\mu) + \delta$ or μ has some structure

Thank You!