§ Tropicalisation

Def A valuation on a field K is the data of an application $\vartheta : K \to \mathbb{R} \cup \{\infty\}$ which verifies:

1. $\vartheta(a) = \infty \iff a = 0$
2. $\vartheta(ab) = \vartheta(a) + \vartheta(b)$
3. $\vartheta(a+b) \geq \min\{\vartheta(a), \vartheta(b)\}$.

In this case K is called a valued field.

Def

$R := \{a \mid \vartheta(a) \geq 0\}$ valuation ring

$\mathfrak{m} := \{a \mid \vartheta(a) > 0\}$ maximal ideal

$k := R/\mathfrak{m}$ residue field

$\vartheta(K^\times) \subseteq \mathbb{R}$ value group

Ex. • $K = \mathbb{Q}$ ϑ_p p-adic valuation

• $K = \mathbb{C}\{\{t\}\}$ the field of Puiseux series

$= \bigcup_{n \in \mathbb{N}} \mathbb{C}((t^{1/n}))$ where $\mathbb{C}((s)) = \left\{ \sum_{i \geq N} a_i s^i \text{ for some } N,\ a_i \in \mathbb{C} \right\}$

field of Laurent series in s.

To simplify $\vartheta(K^\times) \subseteq \mathbb{Q}$ (if K algebraically closed $\vartheta(K^\times) = \mathbb{Q}$)

and we suppose there exists a section to

$$\begin{array}{ccc} K^\times & \xrightarrow{\vartheta} & \vartheta(K^\times) \\ \cup & & \cup \\ t^\omega & \longleftarrow & \omega \end{array}$$

i.e. for all $\omega \in \vartheta(K^\times)$, $t^\omega \in K^\times$

with $\vartheta(t^\omega) = \omega$, $t^{\omega_1 + \omega_2} = t^{\omega_1} \cdot t^{\omega_2}$

<u>Def</u> - $\forall\, a \in R$, $\quad \bar{a} := a \mod \mathfrak{M}$, this is an element of k.

- $\forall\, a \in K^{\times}$ $\quad \tilde{a} := \overline{t^{-v(a)} a}$, this is an element of $k^{\times}$

$\tilde{0} := 0$

- Let $(S, +)$ be a commutative semi-group with $0 \in S$.

⇝ semi-group algebra with K coefficients

$$K[S] := \left\{ \sum_{s \in A} a_s \chi^s \;\middle|\; \begin{array}{l} A \subseteq S \text{ finite and} \\ a_s \in K \;\; \forall\, s \in A \end{array} \right\}$$

the product $\quad \chi^{s_1} \cdot \chi^{s_2} = \chi^{s_1 + s_2}$

- Let M be a free abelian group of finite rank n.

choosing basis $e_1, \ldots, e_n$ of M, we get

$$K[M] \simeq K[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$$

where $X_i := \chi^{e_i}$ in $K[M]$.

<u>Def</u> (Algebraic torus) over K with character group M $\qquad T = T_K := \operatorname{Spec} K[M]$.

The set of K-points of T denoted by $T(K)$ is by definition

$$\operatorname{Mor}_{K\text{-var}}(\operatorname{Spec} K, T_K = \operatorname{Spec} K[M]) = \operatorname{Hom}_{K\text{-alg}}(K[M], K)$$
$$\simeq \operatorname{Hom}(M, K^{\times}).$$

So each K-point x of T corresponds to a morphism of abelian groups (denoted again by x, by an abuse of the notation)

$$x : M \longrightarrow K^\times.$$

<u>Rk.</u> Basis of M gives rise to an identification

$$T(K) = (K^\times)^n.$$

$x \in T(K)$ coordinate $(x_1, \dots, x_n)$. The corresponding map $x : \mathbb{Z}^n \longrightarrow K^\times$ is given by

$$x(m_1, \dots, m_n) = x_1^{m_1} \cdots x_n^{m_n} \quad \text{for all}$$

$$m = m_1 e_1 + \dots + m_n e_n \in M \simeq \mathbb{Z}e_1 \oplus \dots \oplus \mathbb{Z}e_n.$$

<u>Def</u> For each point $x \in T(K)$, the tropicalisation of x is denoted by $\mathrm{trop}(x)$ and is defined by

$$\mathrm{trop}(x) := \nu \circ x : \underbrace{M \xrightarrow{x} K^\times \xrightarrow{\nu} \mathbb{Q}}_{\nu \circ x} \in \mathrm{Hom}(M, \mathbb{Q}).$$

Setting $\underset{\text{dual of } M}{N = M^\vee} = \mathrm{Hom}(M, \mathbb{Z})$, and $\underset{N_{\mathbb{Q}}}{N_{\mathbb{R}}}$ the corresponding real vector space over $\underset{\mathbb{Q}}{\mathbb{R}}$, we get $\mathrm{trop}(x) \in N_{\mathbb{Q}} \subseteq N_{\mathbb{R}}$.

Denote by $\langle \cdot, \cdot \rangle$ the canonical bilinear pairing $N \times M \to \mathbb{Z}$.

Let X be a subvariety of $T = \operatorname{Spec} K[M]$ with defining ideal $I \subseteq K[M]$. So $X = \operatorname{Spec}(K[M]/I)$.

The K-points of X denoted by $X(K)$ are

$$X(K) := \operatorname{Mor}_{K\text{-var}}(\operatorname{Spec} K, X)$$
$$= \{ x \in T(K) \mid f(x) = 0 \quad \text{for all } f \in I \}$$
$$= \{ x : M \to K^* \mid \text{for all } f = \sum a_m x^m \in I, \ \sum a_m x(m) = 0 \}$$

<u>Def</u> Assume K algebraically closed, and $X \subseteq T_K$ an alg. subvariety. The set $\operatorname{Trop}(X)$ called tropicalisation of X is defined by

$$\operatorname{Trop}(X) := \overline{\{ \operatorname{trop}(x) \mid x \in X(K) \}} \subseteq N_{\mathbb{R}}$$

<u>Questions</u>: shape of $\operatorname{Trop}(X)$?
How to calculate $\operatorname{Trop}(X)$?

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* Alternative <u>description</u> of $\operatorname{Trop}(X)$

<u>Lemma</u> Let $a, b \in K$ with $\nu(a) \neq \nu(b)$. Then $\nu(a+b) = \min\{\nu(a), \nu(b)\}$.

<u>Proof</u> We have $\nu(1) = \nu(1) + \nu(1) \Rightarrow \nu(1) = 0$
(since $1.1 = 1$)

$\nu(1) = \nu(-1) + \nu(-1) \Rightarrow \nu(-1) = 0$

Suppose now w.l.g. $\vartheta(a) > \vartheta(b)$.

$$\vartheta(b) \geq \min\{\vartheta(a+b), \vartheta(-a)\} \qquad \text{since } b = a+b+(-a).$$
$$\qquad\qquad\qquad \| $$
$$\min\{\vartheta(a+b), \vartheta(a)\}$$

Since $\vartheta(a) > \vartheta(b) \Rightarrow \vartheta(b) \geq \vartheta(a+b)$.

But $\vartheta(a+b) \geq \min\{\vartheta(a), \vartheta(b)\} = \vartheta(b)$

$\Rightarrow \vartheta(b) = \vartheta(a+b)$. $\square$

This is a direct consequence:

$\boxed{\text{Cor}}$ Suppose $a_1, \dots, a_l \in K^\times$ with $\sum a_i = 0$

$\Rightarrow \min\{\vartheta(a_i)\}$ is achieved at least twice.

$\boxed{\text{Cor}}$ (*) Let $x \in X(K) \subseteq T(K)$, and let $f = \sum a_m \chi^m$ with $f(x) = 0$

$x : M \to K^\times$

$$f(x) = 0 = \sum a_m x(m)$$

$\Rightarrow \min\{\vartheta(a_m) + \langle \text{trop}(x), m\rangle\}$ is achieved at least twice.

$\boxed{\text{Def}}$ - Let $f = \sum a_m \chi^m \in K[M]$. Define $\text{trop}(f) : N_{\mathbb{R}} \to \mathbb{R}$

by setting $\forall\, w \in N_{\mathbb{R}}$,

$$\text{trop}(f)(w) := \min\{\vartheta(a_m) + \langle w, m\rangle\}.$$

- $V(\text{trop}(f)) := \{w \mid \min\{\vartheta(a_m) + \langle w, m\rangle$ is achieved at least twice$\}$

"zero locus" of $V(\text{trop}(f))$

- For an ideal $I \subseteq K[M]$,

$$\Sigma(I) := \bigcap_{f \in I} V(\mathrm{trop}(f)) \subseteq N_{\mathbb{R}}.$$

<u>Rk.</u> By Corollary (*), for any $f \in I$, we have

$$\mathrm{Trop}(X) \subseteq V(\mathrm{trop}(f)).$$

$$\Rightarrow \quad \mathrm{Trop}(X) \subseteq \Sigma(I).$$

<u>Fundamental theorem</u>

For any subvariety X of a torus $T = \mathrm{Spec}\, K[M]$, with defining ideal $I \subseteq K[M]$, we have

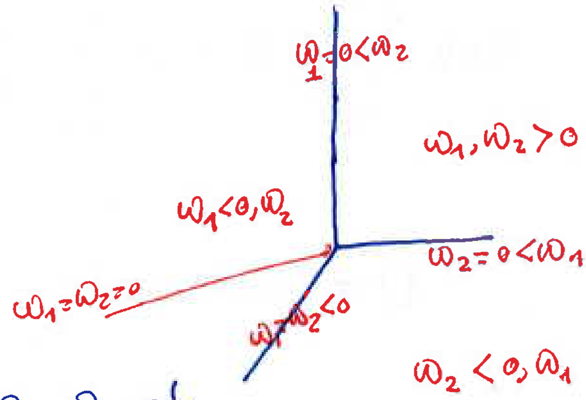
$$\mathrm{Trop}(X) = \Sigma(I).$$

[7]

Examples

(1) $f \in K[X_1^{\pm 1}, X_2^{\pm 1}]$

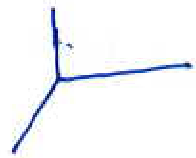
$$f = X_1 + X_2 + 1$$



$$\mathrm{trop}(f) : \mathbb{R}^2 \to \mathbb{R}$$

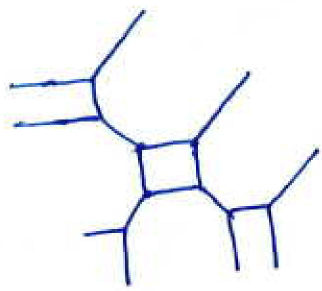
$$(\omega_1, \omega_2) \to \min\{\omega_1, \omega_2, 0\}$$

$\mathrm{trop}((f=0)) =$



(2) Tropicalisation of a curve of degree 3

One of the possibilities



(3) Tropicalisation of a hypersurface

$X = (f=0)$ $\quad f \in K[M]$

$A \subseteq M$ support of f, $\quad f = \sum_{m \in A} a_m x^m$

Newton polytope of f : $P :=$ Convex-hull $\langle m \mid m \in A \rangle$

$$:= \left\{ \sum_{m \in A} \alpha_m m \;\middle|\; \begin{array}{l} \alpha_m \geq 0 \\ \sum \alpha_m = 1 \end{array} \right\} \subseteq M_{\mathbb{R}}.$$

Each a_m gives a heigth to m given by $\nu(a_m)$

This defines a subdivision of P as follows:

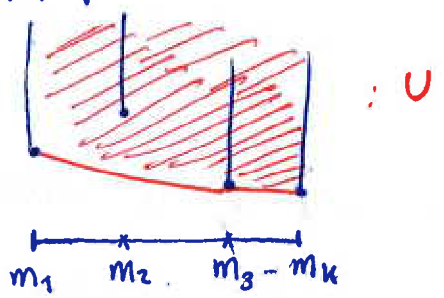
Let $U \subseteq \mathbb{R} \times M_{\mathbb{R}}$ be the "upper-convex-hull" of the height function defined as follows:

$$U = \text{Convex-hull} \langle (\alpha, m) \mid \alpha \geq \vartheta(a_m) ; m \in A \rangle$$

Ex. Suppose M of rank 1

The Newton polytope is an interval

$A = \{m_1, \dots, m_k\}$



$\subseteq \mathbb{R}$

U is a polyhedron (which is rational)

Description of face of U

A face F of U is given by the set of all points where, for a choice of $\alpha^* \in \mathbb{R}$ and $\omega \in N_{\mathbb{R}}$, the linear form $\alpha^* \cdot - + \omega(\cdot)$ is minimised.

I.e $F := \{ (\alpha, m) \in U \mid \alpha^* \alpha + \omega(m)$ is minimised$\}$

The face F is bounded $\Leftrightarrow \alpha^* \neq 0$

If F is defined by (α^*, ω), $\alpha^* \neq 0$) $\Rightarrow \alpha^* > 0$ (otherwise the minimum over U does not exist and the infimum is $-\infty$)

In this case, α^* can be supposed to be equal to 1 (upto changing ω to ω / α^*).

$\Rightarrow$ Finite (bounded) face $F = \{ (\alpha, m) \in U \mid \alpha + \omega(m) \text{ is minimised} \}$

if $(\alpha, m) \in F \Rightarrow \alpha = \vartheta(a_m)$.
$m \in A$

This gives a bijective correspondance

Bounded faces of $U \longleftrightarrow B \subseteq A$ s.t. $\exists\, \omega$ with $\vartheta(a_m) + \langle \omega, m\rangle$ is minimised precisely at elements of B

Under this correspondance

$F_B := \text{Convex-hull} \langle (\vartheta(a_m), m) \rangle_{m \in B} \longleftarrow B$

Bounded face $F \longrightarrow B_F := \{ m \in A \mid (\vartheta(a_m), m) \in F \}$.

Consider now the projection $\pi_2 : U \to P$, $(\alpha, m) \to m$

Bounded faces of U provide a subdivision of P.

$F_B \longrightarrow \text{Convex-hull} \langle m \mid m \in B \rangle \subseteq P$

The subdivisions of a polytope arising in this way are called regular. C.F. Gelfand - Kapranov - Zelvinsky, Discriminants, Resultants, and multidimensional Determinants.

Ex. Let $f \in K[X_1, X_2]$ a polynomial of degree 3.

Two such subdivisions (depending on $\vartheta(\cdot)$ of the coefficients) are the following:

(there are more possibilities)

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the tropicalisation of $\overline{X = (f=0)}$ for $f = \sum_{m \in A} a_m x^m$ has now the following description.

Consider a cell F in subdivision of P with vertex set $B \subseteq A$ and $|B| \geq 2$.

$B \xrightarrow[\text{is associated}]{}$ $N_B \subseteq \mathrm{Trop}(X)$ defined as follows

$$N_B := \left\{ \omega \in N_{\mathbb{R}} \;\middle|\; \begin{array}{l} \vartheta(a_m) + \langle \omega, m \rangle \\ \text{takes its minimum} \\ \text{at points of } B \end{array} \right\}$$

If F has dimension k $\Rightarrow$ N_B has dimension $n-k$

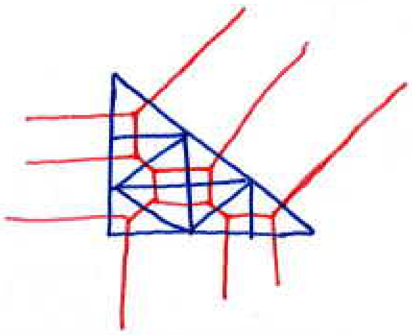
(
$\forall\, m_1, m_2$ we should have
$\vartheta(a_{m_1}) - \vartheta(a_{m_2}) + \langle \omega, m_1 - m_2 \rangle = 0$
)
for $\omega \in N_B$.

In fact N_B is parallel to $F^{\perp}$.

$\Rightarrow$ N_B lives in an affine space of dimension $n-k$.
the other inequalities define N_B in this space

$\Rightarrow$ $\mathrm{Trop}(X)$ is the dual to the subdivision of P and $\{N_B \mid |B| \geq 2\}$ provide a subdivision of $\mathrm{Trop}(X)$.

Ex.



* Hyperplane arrangements

Let $H_1, \dots, H_m \subseteq \mathbb{P}^n_{\mathbb{C}}$ hyperplane where each H_i is the zero set of a linear form $l_i = \sum_{j=0}^{n} a_{ij} z_j$.

Let $X := \mathbb{P}^n_{\mathbb{C}} \setminus \bigcup_{i=1}^{m} H_i$

We have a map $X \longrightarrow \mathbb{C}^{\times m}/\mathbb{C}$ (Torus T with character lattice $M = \{(m_1, \dots, m_m) \in \mathbb{Z}^m \mid \sum m_j = 0\}$ and $N = \mathbb{Z}^m/\mathbb{Z}(1, \dots, 1)$)

$x \longrightarrow (l_1(x), \dots, l_m(x))$

If l_i generate the whole space of linear forms we get embedding

$X \hookrightarrow T = \mathrm{Spec}\, \mathbb{C}[M]$.

Equations of X

$\{l_i\}_{i\in A}$ for $A \subseteq \{1,\dots,m\}$ dependent

i.e. s.t. $\sum_{i\in A} p_i l_i = 0$ gives a linear equation

$$\sum p_i X_i = 0 \quad \text{in } T$$

$$\mathrm{Trop}(X) = \left\{ (\omega_1,\dots,\omega_m) \in N_{\mathbb{R}} = \mathbb{R}^m/\mathbb{R}(1,\dots,1) \;\middle|\; \begin{array}{l} \text{for all } A \subseteq \{1,\dots,m\} \\ \text{dependent,} \\ \min_{i\in A}\{\omega_i\} \text{ is} \\ \text{achieved at least} \\ \text{twice} \end{array} \right\}$$

Let $\underset{\substack{\shortparallel \\ (\omega_1,\dots,\omega_m)}}{\omega} \in \mathrm{Trop}(X) \subseteq \mathbb{R}^m/\mathbb{R}(1,\dots,1)$. We can assume $\min\{\omega_1,\dots,\omega_m\} = 0$

Then writing $\omega = \alpha_1 1_{F_1} + \alpha_2 1_{F_2} + \dots + \alpha_k 1_{F_k}$

for unique $\emptyset \neq F_1 \subsetneq F_2 \subsetneq \dots \subsetneq F_k \subsetneq E$ and $\alpha_1,\dots,\alpha_k > 0$

then we see that each F_i is "closed" in the sense that an element $1\le j\le m$ s.t. l_j is dependent to elements l_h with $h\in F_i$ is already in F_i.

Proof that F_i is closed

Suppose this is not the case. Let l_j dependent to F_i so $l_j = \sum_{h\in F_i} p_h l_h$

Each $h\in F_i$ we have $\omega_h \geq \alpha_i + \alpha_{i+1} + \dots + \alpha_k$ $\Big) \Rightarrow \min_{h\in F_i}\{\omega_j, \omega_h\}$ is achieved only once. X.

while $\omega_j \leq \alpha_{i+1} + \dots + \alpha_k$

Trop(X) has a fan structure where the cones are of the form $\sigma_{\mathcal{F}} = \mathbb{R}_+ 1_{F_1} + \dots + \mathbb{R} 1_{F_k}$ for $k \in \mathbb{N} \cup \{0\}$ and $\mathcal{F} = \{F_1 \subsetneq \dots \subsetneq F_k \subsetneq E\}$ a flag of non-empty closed sets in $\{1, \dots, m\}$.

This is called the Bergman fan of the hyperplane arrangement. For more details see

Ardila-Klivans, the Bergman complex of a matroid and phylogenetic trees.

§ Proof of the fundamental theorem

Reference: See Maclagan-Sturmfels, Introduction to Tropical Geometry and references there.

K non-trivially valued field

$\overline{K} = K$ (algebraically closed)

and to simplify, assume $\vartheta(K^\times) = \mathbb{Q}$.

$X \subseteq T = \operatorname{Spec} K[M]$ given by the ideal $I \subseteq K[M]$.
subvariety

$$\boxed{\text{Fundamental thm} \quad \operatorname{Trop}(X) = \Sigma(I).}$$

where $\Sigma(I) = \bigcap_{f \in I} V(\operatorname{trop}(f))$.

Recall

$$\tilde{a} = \overline{t^{-\nu(a)} a} \in k$$

Def for $\omega \in N_{\mathbb{Q}}$ and $f \in K[M]$, the initial part of f with respect to ω denoted by $\operatorname{in}_\omega(f)$ is defined by

$$\operatorname{in}_\omega(f) = \sum_{\substack{m \\ \text{with } \vartheta(a_m) + \langle \omega, m\rangle \\ = \operatorname{trop}(f)(m)}} \tilde{a}_m x^m.$$

(I.e only those terms with $\nu(a_m) + \langle \omega, m \rangle$ minimum contribute)

Remarks ① $\omega \in V(\operatorname{trop}(f)) \Leftrightarrow \operatorname{in}_\omega(f)$ not a monomial

② $\omega \in \Sigma(I) \Leftrightarrow \operatorname{in}_\omega(f)$ not a monomial for all $f \in I$

$\Leftrightarrow \operatorname{in}_\omega(I) \neq \langle 1 \rangle = k[M]$.

Here $in_\omega(I) :=$ the ideal generated by $in_\omega(f)$ for $f \in I$.

③ For $f, g \in K[M]$, $in_\omega(fg) = in_\omega(f)\, in_\omega(g)$.

Proof of the fund. thm. We need to show $\Sigma(I) \cap N_{\mathbb{Q}} \subseteq Trop(X)$.

By induction on codimension of X in T.

First case Suppose X is a hypersurface given by $f = 0$ for $f \in K[M]$.

$I = \langle f \rangle$

In this case proceed by induction on $n = rank(M)$.

① $n = 1$. $f \in K[X, X^{-1}]$.

Let $\omega \in \Sigma(I) \cap N_{\mathbb{Q}}$. We need to show that $\omega \in Trop(X)$.

$\omega \in \Sigma(I) \Rightarrow in_\omega(f)$ is not a monomial.

Write $f = X^a \prod_{j=1}^{b} (a_j X - b_j)$

with $a, b \in \mathbb{Z}$, $b \geq 0$ $a_j, b_j \in K$

$\Rightarrow in_\omega(f) = X^a \prod_{j=1}^{b} in_\omega(a_j X - b_j)$

$\Rightarrow \exists j$ with $in_\omega(a_j X - b_j)$ not a monomial

$\Rightarrow v(a_j) + \omega = v(b_j)$

Let $x = b_j / a_j$. We have $f(x) = 0$, $v(x) = \omega$

$\Rightarrow \omega = trop(x) \in Trop(X)$.

② General $n = \underline{rk(M)}$ assuming $n-1$.

Let $\omega \in \Sigma(I) \cap N_{\mathbb{Q}}$, so $in_\omega(f) \neq 0$

$\Rightarrow$ $\exists$ $y : M \to k^\times$ such that $in_\omega(f)(y) = 0$

<u>claim</u> $\exists$ primitive element $l \in M$ and decomposition

$M = \mathbb{Z}l \oplus M'$ for M' of rank $n-1$

s.t. $\quad in_\omega(f)\Big|_{x^l = y(l)} \neq 0$.

For a polynomial $F = \sum b_m x^m$, and α in the field

define $\quad F\Big|_{x^l=\alpha} := \sum_{\substack{m=(al,m') \\ \in M = \mathbb{Z}l \oplus M'}} b_m \alpha^a x^{m'}$.

Note that $x^m = (x^l)^a x^{m'}$

and we give value α to x^l.

<u>Proof of the claim</u>

Let $A \subseteq M$ be the support of $in_\omega(f)$, and

let $l \in M$ be a primitive vector s.t. l is not

parallel to $m_1 - m_2$ for any pair of distinct elements

$m_1, m_2 \in A$. l exists since A is finite.

Then projections of elements of A to M' under

$\pi_2 : M \to M'$

are all distinct.

$\Rightarrow$ $in_\omega(f)\Big|_{x^l = y(l)} := \sum_{\substack{m=(al,m') \\ \in A}} \tilde{a}_m \alpha^a x^{m'}$ has support $\pi_2(A)$. $\square$

Let now α be an element of $K^\times$ with $\vartheta(\alpha) = \omega(\ell)$ and $\tilde{\alpha} = y(\ell)$. Let $\omega' := \omega|_{M'}$, $y' = y|_{M'}$, and $g = f|_{x^\ell = \alpha}$.

We have ① $in_{\omega'}(g) = in_\omega(f)|_{x^\ell = y(\ell)}$

② $in_{\omega'}(g)(y') = 0$

in particular, since $in_{\omega'}(g) \neq 0$, ② implies that $in_{\omega'}(g)$ is not a monomial.

By induction, $\exists x' : M' \to K^\times$ with $trop(x') = \omega'$ and $g(x') = 0$.

Let $x : \underset{\overset{\|}{M}}{\mathbb{Z}\ell \oplus M'} \to K^\times$ be defined by $x(\ell) = \alpha$

$x|_{M'} = x'$.

We have $f(x) = g(x') = 0$, and $trop(x) = \omega$, as required.

2nd Case $X \subseteq T$ of higher codimension

The proof in this case goes by dimension reduction to the case of a hypersurface.

For this we need to consider morphisms of tori.

Let $\varphi^\#: M' \to M$ be a morphism of abelian groups

This induces a morphism of K-algebras $K[M'] \to K[M]$

and a morphism $\varphi: T \to T'$ of tori.

The morphism $\varphi^\#$ induces by duality a morphism

$\varphi: N \longrightarrow N'$ of dual lattices. ($\varphi = (\varphi^\#)^\vee$)

Let $I \subseteq K[M]$ an ideal and $I' = (\varphi^\#)^{-1}(I) \subseteq K[M']$.

<u>Prop</u> • The map $\phi: N \to N'$ restricts to a morphism $\Sigma(I) \to \Sigma(I')$.

• Let $\underset{T}{\overset{X}{\cap}}$ and $\underset{T'}{\overset{X'}{\cap}}$ be the subvarieties of T and T' associated to I and I'. φ restricts to a morphism $\mathrm{Trop}(X) \longrightarrow \mathrm{Trop}(X')$.

<u>Proof</u> By direct verification. For the first one, one verifies for $\omega \in N_{\mathbb{R}}$ if $in_\omega(I) \neq \langle 1 \rangle \Rightarrow$ denoting $\omega' = \varphi(\omega)$ then $in_{\omega'}(I') \neq \langle 1 \rangle$. This shows $\omega' \in \Sigma(I')$.

The reduction to the hypersurface case is by:

<u>Technical Prop</u> Let $X \subseteq \underset{\mathrm{Spec}\,K[M]}{T}$ a subvariety of dimension d. with ideal I of definition

Let $\omega \in \Sigma(I) \cap N_{\mathbb{Q}}$. Then

$\exists$ an injective $\varphi^\#: M' \longrightarrow M$ with M' of rank $d+1$ inducing a morphism of tori $\varphi: T \longrightarrow T'$ with $T' = \mathrm{Spec}(M')$

s.t.
1. $\dim(T') = d+1$ ($=$ rank M', this is automatic)
2. $\varphi(X)$ is a closed subvariety of T'.
3. $\dim \varphi(X) = d$
4. ω is the only point of $\Sigma(I)$ with image equal to $\omega' := \varphi(\omega)$. In other words, $\varphi^{-1}_{|\Sigma(I)}(\varphi(\omega)) = \omega$.

Proof of the fundamental thm assuming the Proposition

In order to prove $\mathrm{Trop}(X) = \Sigma(I)$, it will be enough to show that any $\omega \in \Sigma(I) \cap N_{\mathbb{Q}}$ belongs to $\mathrm{Trop}(X)$, i.e. $\exists\ x \in X(K)$ with $\mathrm{trop}(x) = \omega$.

By applying the previous proposition, we find an application of tori $\varphi : T \to T'$ with $\dim(T') = d+1$, and $X' = \varphi(X)$ a closed subvariety of T' of dim d, and $\varphi|_{\Sigma(I)}^{-1}(\omega') = \{\omega\}$ where $\omega' := \varphi(\omega)$.

Let I' the defining ideal of X'. By the case of hypersurfaces, we have $\mathrm{Trop}(X') = \Sigma(I')$. Since $\omega' \in \Sigma(I')$, we find a pt $x' \in X'(K)$ with $\mathrm{trop}(x') = \omega'$. Since $X(K) \to X'(K)$ is surjective, $\exists\ x \in X(K)$ with ~~trop(x)~~ $\varphi(x) = x'$. In particular $\mathrm{trop}(x)$ is mapped to $\omega' = \mathrm{trop}(x')$. Since $\mathrm{trop}(x) \in \Sigma(I)$, by the last property of the map φ, we find $\mathrm{trop}(x) = \omega$. $\square$

$$
\begin{tikzcd}
X(K) \ni x \arrow[r, two heads] \arrow[d] & X'(K) \ni x' \arrow[d] \\
\mathrm{Trop}(X) \subseteq \Sigma(I) \arrow[r] & \mathrm{Trop}(X') = \Sigma(I') \\
\mathrm{trop}(x) \arrow[r] & \omega'
\end{tikzcd}
$$

$\Rightarrow \mathrm{trop}(x) = \omega$ since $\varphi|_{\Sigma(I)}^{-1}(\omega') = \{\omega\}$.

Properties of the polyhedral structure on $\Sigma(I)$

Let $I \subseteq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and $\Sigma(I) \subseteq \mathbb{R}^n$ with corresponding polyhedral str.

① $\Sigma(I)$ has dimension $d = \dim X$

② X irreducible $\Rightarrow$ $\Sigma(I)$ is pure of dim d
i.e. all maximal faces of $\Sigma(I)$ have dim d.

③ ω_1, ω_2 in the relative interior of the same face of $\Sigma(I)$, $in_{\omega_1}(I) = in_{\omega_2}(I)$

④ Local structure: for all $\omega \in \Sigma(I)$

$$Star(\omega) = \Sigma(in_\omega(I))$$

$$\left(\begin{aligned} Star(\omega) &:= \{ z \mid \omega + \varepsilon z \text{ in } \Sigma(I) \text{ for } \varepsilon > 0 \text{ suff small} \} \\ &= \{ z \mid in_{\omega+\varepsilon z}(I) \neq \langle 1 \rangle \text{ for } \varepsilon > 0 \text{ suff. small} \} \\ &= \{ z \mid in_z(in_\omega(I)) \neq \langle 1 \rangle \quad \text{" " " "} \} \\ &= \Sigma(in_\omega(I)) \end{aligned} \right.$$

Proof of the technical proposition

Consider $\Sigma(I)$ with a polyhedral structure.

Let $w \in \Sigma(I) \cap N_{\mathbb{Q}}$. For each face σ, of $\Sigma(I)$

Let L_σ be the vector space generated by $w - u$ for $u \in \sigma$.

① $\dim(L_\sigma) \leq d+1$

② $\{L_\sigma\}_{\sigma \text{ face of } \Sigma(I)}$ is finite.

Claim Let $X \subseteq T = \operatorname{Spec} K[M]$ of dimension d.
$\exists$ M' of rank $d+1$ and an injection $\varphi^\#: M' \longrightarrow M$ s.t. for the map $\varphi: T \to T' = \operatorname{Spec} K[M']$

① $\varphi(X)$ is closed in T'

② $\dim \varphi(X) = d$

③ $\ker(\varphi: N \longrightarrow N')$ intersects trivially a fixed collection $H_1, H_2, \ldots H_k$ of subspace of $N_{\mathbb{R}}$ with $\dim(H_i) \leq d+1$.

Proof of the technical Prop Apply the claim to the collection $\{L_\sigma\}$. to find $\varphi: T \to T'$ with $\ker(\varphi: N \to N') \cap L_\sigma = (0)$.

①, ②, ③ in the prop follow from the properties of φ.

To prove ④ let $u \in \Sigma(I)$ with $\varphi(u) = \varphi(w)$.

Let σ a face of $\Sigma(I)$ containing u.

$\Rightarrow$ $w - u \in L_\sigma$
$w - u \in \ker(\varphi)$ $\Rightarrow$ $w - u \in L_\sigma \cap \ker(\varphi) = (0)$

$\Rightarrow$ $\underline{w = u}$.

Proof of the claim

By Induction.

Let I the ideal of X and pick $f = \sum_{\substack{m \in A \\ m \neq 0}} a_m x^m \in I$

for $A \subset M$ finite with $f \neq 0$.

We look first for a decomposition $M = \mathbb{Z}l \oplus M_1$

s.t. the projection $\pi_1 : M \longrightarrow \mathbb{Z}l$ restricted to A

be injective. For this it will be enough to have $m_1 - m_2 \notin M_1$

for all pairs of distinct elements $m_1, m_2 \in A$. We can choose

l and M_1 with this property.

Write $f = \sum_{\substack{m = (al, m_1) \\ m \in A}} a_m (x^l)^a x^{m_1}$

Let $\varphi_1^\# : K[M_1] \to K[M]$ and $I_1 = \varphi_1^{\#-1}(I)$

Let $A_1 := K[M_1]/I_1 \longleftarrow K[M]/I =: A$.

Since x^l verifies $f = 0$ with coefficients in $K[M_1]$ which are all monomials, It follows that A is finite over A_1

$\Rightarrow$ $\operatorname{Spec} A \longrightarrow \operatorname{Spec} A_1$ is finite

and so $\varphi_1(X) = X_1 = \operatorname{Spec} A_1$ is a closed subvariety

of $T_1 = \operatorname{Spec} K[M_1]$ of dimension $= \dim X$.

We need to show that we can choose M_1 and l

s.t. $\operatorname{Ker}(\varphi_1 : N \to N_1)$ be not contained in any of the

H_i's for $i = 1, \to k$.

(Then an induction allows to finish the proof)

Let τ be a primitive element of N with

$\tau \notin \bigcup_{i=1}^{k} H_i$ and s.t $\tau(m_1 - m_2) \neq 0$ for $m_1 \neq m_2$ in A.

Let $M_1 = \operatorname{Ker}(\tau)$ and choose l primitive with

$$M = \mathbb{Z}l \oplus M_1.$$

Since M_1 does not contain $m_1 - m_2$ for $m_1 \neq m_2$ in A $\Rightarrow$ we can apply the reasoning above to get a map $\varphi_1 : T \to T_1$

Since $\tau \notin H_i \Rightarrow \operatorname{Ker}(\varphi_1 : N_{\mathbb{R}} \to N_{1,\mathbb{R}}) = \mathbb{R}\tau \notin H_i$,

and the claim follows.

§ Toric varieties (and connection to tropicalisations)

Reference

- Kempf - Knudsen - Mumford - Saint-Donat, Toroidal Embeddings I.
- Fulton, Introduction to Toric Varieties
- Tevelev, Compactifications of subvarieties of tori
- Maclagan - Sturmfels, Introduction to tropical geometry

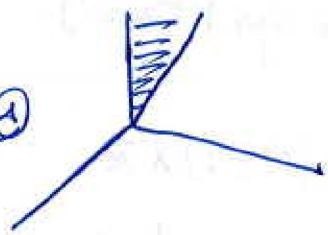
Let M and N be free abelian groups of finite rank n, as before with $N = M^\vee = \mathrm{Hom}(M, \mathbb{Z})$.

Def A (rational) fan Σ in $N_{\mathbb{R}}$ is a finite collection of rational strict convex cones $\sigma \subseteq N_{\mathbb{R}}$ verifying

(1) if $\sigma \in \Sigma$ and τ a face of σ $\Rightarrow$ $\tau \in \sigma$.

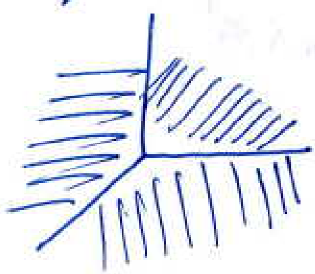
(2) for $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a common face of σ_1 and σ_2 (and thus belongs to Σ).

Ex. (1)



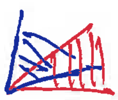
Σ has
1 cone of dim two
4 cones of dim one
1 cone of dim 0

(2)



Σ has
3 cones of dim. two
3 " " " one
1 cone of dim zero.

(3) Non-example



For $\sigma \in \Sigma$, $\quad \sigma^\vee := \left\{ m \in M_{\mathbb{R}} \;\middle|\; \begin{array}{l} \langle \omega, m\rangle \geq 0 \\ \forall\, \omega \in \sigma \end{array} \right\}$

the dual cone.

$\sigma^\vee$ is a rational cone with $\dim \sigma^\vee = n$ (since σ is strict)

(strict means there is no line in σ)

Define $\quad S_\sigma := \sigma^\vee \cap M \quad$ the integer pts of $\sigma^\vee$.

Gordon's <u>Lemma</u> $\quad S_\sigma$ is a finitely generated monoid (with addition).

<u>Def</u> $\quad K[S_\sigma]$ the semi-group algebra with coefficients in K

$U_\sigma := \operatorname{Spec} K[S_\sigma]$ the corresponding algebraic variety.

(Here K is any field)

<u>Ex.</u>

• 

$S_\sigma \cong (\mathbb{N} \cup \{0\})^2$

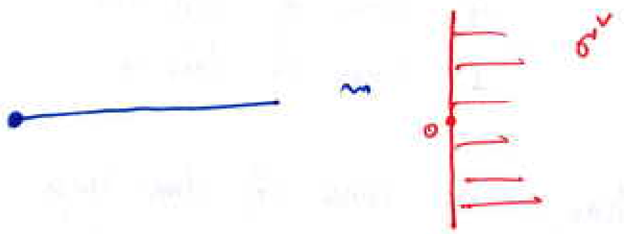
$U_\sigma \simeq \mathbb{A}^2_K$

• 

$S_\sigma = M \simeq \mathbb{Z}^2$

$U_\sigma \simeq \mathbb{G}_m \times \mathbb{G}_m$

$\mathbb{G}_m := \operatorname{Spec} K[x, x^{-1}]$.

• 

$S_\sigma = (\mathbb{N} \cup \{0\}) \times \mathbb{Z}$

$U_\sigma \simeq \mathbb{G}_m \times \mathbb{A}^1$

Rk. $\mathrm{Hom}_{\text{alg.group}}(T, \mathbb{G}_m) = M$ $\quad$ M is the group of characters of T

$\mathrm{Hom}_{\text{alg.group}}(\mathbb{G}_m, T) = N$ $\quad$ N is the group of cocharacters

For $\omega \in N$, $\lambda_\omega : \mathbb{G}_m \to T$ the corresponding map, which is given by the map $K[M] \to K[\mathbb{Z}]$ induced by $\omega : M \to \mathbb{Z}$. (The map sends $\chi^m \to \chi^{\langle \omega, m \rangle}$.)

Rk. $\sigma \subseteq N_{\mathbb{R}}$ is characterised as follows:

σ is the convex-hull of all $\omega \in N$ s.t.

$\lambda_\omega : \mathbb{G}_m \to T$ extends to a morphism $\mathbb{A}^1 \to U_\sigma$,

i.e. $\lim_{t \to 0} \lambda_\omega(t)$ exists in U_σ.

($\omega \in \sigma \cap N \iff$ for all $m \in S_\sigma$, $\langle \omega, m \rangle \geq 0$

$\iff \lambda_\omega^\# : K[M] \to K[\mathbb{Z}]$ induces

$K[S_\sigma] \to K[\mathbb{Z}_{\geq 0}]$ i.e. $\mathbb{A}^1 \to U_\sigma$.)

Σ a fan in $N_{\mathbb{R}}$,

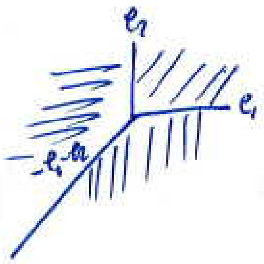
$|\Sigma| :=$ support of $\Sigma = \bigcup_{\sigma \in \Sigma} \sigma$.

$\tau \leq \sigma \Rightarrow U_\tau \hookrightarrow U_\sigma$ open subvariety

$\sigma^\vee \subseteq \tau^\vee$ and $S_\tau = S_\sigma[-m]\big|_{m \in \sigma^\vee \cap \tau^\perp \cap M}$

$\Rightarrow K[S_\sigma] \longrightarrow K[S_\tau]$ is a localisation inducing an open embedding $U_\tau \hookrightarrow U_\sigma$.

Def $\mathbb{P}_\Sigma := \bigsqcup_{\sigma \in \Sigma} U_\sigma \big/ \sim$ the variety obtained by gluing U_σ under the identification of U_τ as an open subset of U_σ for $\tau \leq \sigma$.

Ex. 

$\rightsquigarrow \mathbb{P}_\Sigma \simeq \mathbb{P}^2$

Rk. The actions of T on U_σ are compatible and give an action of T on $\mathbb{P}_\Sigma$.

Properties

① $\mathbb{P}_\Sigma$ is separated.

② $\mathbb{P}_\Sigma$ is normal. In fact any normal variety $\mathbb{P} \supseteq T$ which admits a compatible action by T, i.e.

$$\begin{array}{ccc} T \times \mathbb{P} & \longrightarrow & \mathbb{P} \\ \cup & \square & \cup \\ T \times T & \longrightarrow & T \end{array}$$

is of the form $\mathbb{P} = \mathbb{P}_\Sigma$ for some fan Σ in $N_{\mathbb{R}}$.

③ Stratification in torus orbits:

For $\sigma \in \Sigma$, the dual cone $\sigma^\vee$ contains

$$\sigma^\vee \supseteq \sigma^\perp := \{ m \in M_{\mathbb{R}} \mid \langle \omega, m \rangle = 0 \ \forall \omega \in \sigma \}.$$

Define $M_\sigma := \sigma^\perp \cap M$.

$$\operatorname{rank}(M_\sigma) = \operatorname{codim}(\sigma) = n - \dim(\sigma).$$

We have a morphism of semi-groups:

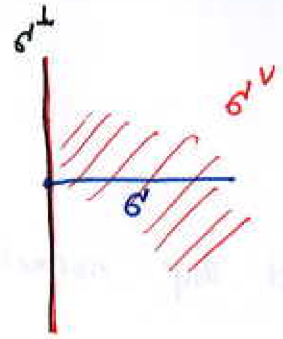
$$S_\sigma \longrightarrow M_\sigma$$
$$m \longrightarrow \begin{cases} m & \text{if } m \in M_\sigma \\ 0 & \text{otherwise.} \end{cases}$$

This gives an embedding of $T_\sigma := \operatorname{Spec}(\mathbb{C}[M_\sigma])$ in U_σ,

$$T_\sigma \xhookrightarrow[\text{closed}]{} U_\sigma \qquad \dim(T_\sigma) = \operatorname{codim}(\sigma).$$

We have $\mathbb{P}_\Sigma = \bigsqcup_{\sigma \in \Sigma} T_\sigma$.

Ex.



$$U_\sigma \simeq \mathbb{A}^1 \times \mathbb{G}_m$$
$$\cup\!\!\mid$$
$$T_\sigma = \{0\} \times \mathbb{G}_m$$

Ex.

$$\Sigma = \{ (0), \mathbb{R}_{\geq 0} e_1, \mathbb{R}_{\geq 0} e_2, \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \}$$

$$\mathbb{A}^2 = \mathbb{G}_m \times \mathbb{G}_m \sqcup \{0\} \times \mathbb{G}_m \sqcup \mathbb{G}_m \times \{0\} \sqcup \{ \underset{(0,0)}{\overset{\shortparallel}{0}} \}.$$

Define $V_\sigma := \overline{T_\sigma}$ the closure of T_σ in $\mathbb{P}_\Sigma$.

We have $V_\sigma = \bigcup_{\substack{\eta \in \Sigma \\ \text{with } \eta \supseteq \sigma}} T_\eta$.

(4) $\mathbb{P}_\Sigma$ is complete $\Longleftrightarrow$ Σ is complete, i.e. $|\Sigma| = N_{\mathbb{R}}$.

More generally, <u>Prop</u> Let $X \subseteq T$ a closed subvariety, $K = k$ trivially valued.

the closure $\overline{X}$ of X in $\mathbb{P}_\Sigma$ is complete

$\Longleftrightarrow$ $|\Sigma| \supseteq \mathrm{Trop}(X)$.

<u>Proof</u> Valuative criterium of properness

Let L be a valued field $\supseteq k$, with valuation ring R, let $x : \mathrm{Spec}\, L \to X$ an L-point of X. We need to show the existence of a map $\mathrm{Spec}\, R \to \overline{X}$ extending x.

$x \in X(L) \Rightarrow \quad \mathrm{trop}(x) \in \mathrm{Trop}(X) \subseteq |\Sigma|$ by assumption.

$\Rightarrow \exists\ \sigma \in \Sigma$ with $\mathrm{trop}(x) \in \sigma$.

$x \in X(L) \subseteq T(L)$ is given by a map

$$x : k[M] \longrightarrow L^{\times}$$

$$x : M \longrightarrow L^{\times}$$

and $\underset{v \circ x}{\overset{\shortparallel}{\mathrm{trop}(x)}} : M \longrightarrow \mathbb{R}$

$\mathrm{trop}(x) \in \sigma \Rightarrow \quad \forall\ m \in S_\sigma \subseteq \sigma^{\vee}, \quad \langle \mathrm{trop}(x), m \rangle \geq 0$

$\Rightarrow \quad \mathrm{trop}(x) : S_\sigma \longrightarrow \mathbb{R}_{\geq 0}$ and so $v(x(m)) \geq 0 \ \forall m \in S_\sigma$ i.e. $x(m) \in R$

$\Rightarrow \quad x : k[S_\sigma] \longrightarrow R$

$\Rightarrow$ induce a map $\mathrm{Spec}\, R \longrightarrow U_\sigma \subseteq \mathbb{P}_\Sigma$.

This map factorise through $\bar{X}$. Thus we get

$$\begin{tikzcd} \bar{X} & \mathrm{Spec}\, R \arrow[l] \\ X \arrow[u, hook] & \mathrm{Spec}\, K \arrow[l] \arrow[u, hook] \end{tikzcd}$$

extending x. $\square$

$\underline{\text{Rk.}}$ This shows Tropicalisation $\longleftrightarrow$ Compactification over k (trivially valued case)

More $\underline{\text{generally}}$ over non-trivially valued field: tropicalisation $\longleftrightarrow$ compactification over K + model of the compactification over $\mathrm{Spec}\, R$ (toric schemes)

Balancing Condition and intersections in $\mathbb{P}_\Sigma$

Σ a fan, $\mathbb{P}_\Sigma$ corresponding toric variety.

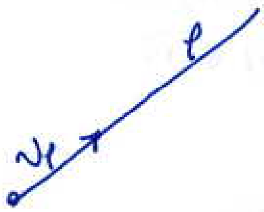
$\forall m \in M$, χ^m is a rational function on $\mathbb{P}_\Sigma$.

<u>Question</u> Let $\rho \in \Sigma$ be a ray, i.e. a cone of dim one.

Let V_ρ be the corresponding closed subvariety of codim one in $\mathbb{P}_\Sigma$.

$$\begin{array}{c} V_\rho \\ \cup \\ T_\rho \end{array}$$

$$\mathrm{ord}_{V_\rho}(\chi^m) = \,?$$



Let v_ρ be a primitive vector in ρ.

$$N = \mathbb{Z}v_\rho \oplus N'$$

$$M = \mathbb{Z}l \oplus M' \quad \text{for} \quad M' = \rho^\perp \cap M = M_\rho$$

and $l \in M$ with $\langle v_\rho, l\rangle = 1$.

$$\begin{array}{ccc} U_\rho & \simeq & \mathbb{A}^1 \times T_\rho \\ \cup & & \cup \\ T_\rho & = & \{0\} \times T_\rho \end{array}$$

with $\mathbb{A}^1 \cong \mathrm{Spec}\, K[\mathbb{Z}_{\geq 0} l]$

$T_\rho = \mathrm{Spec}\, K[M_\rho]$.

$m \in M$ can be written as $al + m'$ with $a \in \mathbb{Z}$ and $m' \in M' = M_\rho$.

$$\mathrm{ord}_{T_\rho}(\chi^m) = \mathrm{ord}_0(\chi^{al}) = a = \langle v_\rho, m\rangle.$$

$$\Rightarrow \quad \boxed{\mathrm{ord}_{V_\rho}(\chi^m) = \langle v_\rho, m\rangle}$$

$$\Rightarrow \quad \mathrm{div}(\chi^m) = \sum_{\substack{\rho \in \Sigma \\ \text{ray}}} \langle v_\rho, m \rangle V_\rho$$

More generally, let $\tau \in \Sigma$, and $m \in M_\tau \; (= \tau^\perp \cap M)$.

χ^m defines a rational function on V_τ and we have

$$\boxed{\mathrm{div}(\chi^m) = \sum_{\substack{\Sigma \ni \sigma' \supsetneq \tau \\ \dim(\sigma') = \dim(\tau)+1}} \langle v_{\sigma'/\tau}, m \rangle V_{\sigma'}}$$

where $v_{\sigma'/\tau}$ is a primitive vector in N with $N_{\sigma'}$ generated by N_τ and $v_{\sigma'/\tau}$ and $v_{\sigma'/\tau}$ "pointing into the interior of σ'"

i.e. $\forall$ ω in the relative interior of τ, $\tau + \varepsilon v_{\sigma'/\tau}$ is in σ' for $\varepsilon > 0$ sufficiently small.

Here $N_\eta = (\text{Vector space generated by } \eta) \cap N$ for all $\eta \in \Sigma$.

Let now $K = \overline{K}$ valued field.

$X \subseteq T_K$ a closed subvariety with defining ideal I.
Suppose X irreducible. $\dim(X) = d$.
Let Σ be a polyhedral structure on $\mathrm{Trop}(X) = \Sigma(I)$

verifying for all polyhedron $\sigma \in \Sigma$,
for ω_1, ω_2 in the relative interior of σ,
$in_{\omega_1}(I) = in_{\omega_2}(I)$.

Define m: d-dimensional cells of Σ $\longrightarrow \mathbb{N}$

as follows:

Let $\sigma \in \Sigma$ d-dim cell.

$\omega \in$ rel. int of σ.

let $A := k[M]/in_\omega(I)$ and $\mathfrak{p}_1, \ldots, \mathfrak{p}_\ell$ the minimal prime ideals of A.

$$m(\sigma) := \sum_{i=1}^{\ell} \mathrm{length}(A_{(\mathfrak{p}_i)}).$$

Let τ a $(d-1)$-dim polyhedron in Σ
and σ a d-dim polyhedron in Σ containing τ.
Define $v_{\sigma/\tau}$, as before, as a primitive vector in N
s.t. ① N_σ is generated by N_τ and $v_{\sigma/\tau}$;
② $v_{\sigma/\tau}$ "points into the interior of σ".

Here N_η is the integer points of the vector space generated by all points $\omega_1 - \omega_2$ for $\omega_1, \omega_2 \in \eta$.

Thm (Balancing condition)

Notation as above, $\mathrm{Trop}(X)$ with the weight function m: d-dimensional cells $\to \mathbb{N}$ verify the following "balancing condition"

$\forall\ \tau \in \Sigma$ of dimension $d-1$,

$$\sum_{\substack{\Sigma \ni \sigma' \supsetneq \tau \\ \dim(\sigma') = d}} m(\sigma')\, v_{\sigma'/\tau} \in N_\tau .$$

Rk. This can be regarded as a tropical version of the Poincaré-Lelong formula in complex geometry.

Proof of the thm (sketch)

We can work in $\mathrm{Star}(\tau)$ which is $\mathrm{Trop}(Y)$ for $Y \subseteq T_K = \mathrm{Spec}\, k[M]$ given by ideal $\mathrm{in}_\omega(I) \subseteq k[M]$ for ω in the rel. int of τ.

(Recall $\mathrm{Star}(\tau) := \{ z \in N_{\mathbb{R}} \mid \omega + \varepsilon z \in \mathrm{Trop}(X)$ for $\varepsilon > 0$ sufficiently small $\}$.)

Σ a fan structure on $\mathrm{Star}(\tau)$

Let $\mathbb{P}_\Sigma$ corresponding toric variety.

For $\sigma \in \Sigma$ of dimension d, $V_\sigma \subseteq \mathbb{P}_\Sigma$ has codim d.

and we have

$$m(\sigma) = \overline{Y} \cdot V_\sigma$$

where $\overline{Y}$ is the closure of Y in $\mathbb{P}_\Sigma$ (which is complete).

The balancing condition is the statement that

for τ of dimension $d-1$ in Σ,

$$\sum_{\substack{\sigma' \supsetneq \tau \\ \dim(\sigma') = d}} m(\sigma') \, v_{\sigma'/\tau} \in N_\tau$$

$$\Updownarrow$$

By duality

$$\forall m \in M_\tau \qquad \left\langle \sum_{\sigma' \supsetneq \tau} m(\sigma') \, v_{\sigma'/\tau}, m \right\rangle = 0$$

But

$$\left\langle \sum_{\sigma' \supsetneq \tau} m(\sigma') \, v_{\sigma'/\tau}, m \right\rangle = \left\langle \sum_{\sigma' \supsetneq \tau} \overline{Y} \cdot V_{\sigma'} \, v_{\sigma'/\tau}, m \right\rangle$$

$$= \overline{Y} \cdot \left(\sum_{\sigma' \supsetneq \tau} \langle v_{\sigma'/\tau}, m \rangle V_{\sigma'} \right)$$

$$= (\overline{Y} \cap V_\tau) \cdot \operatorname{div}(\chi^m) \longleftarrow \text{in } V_\tau$$

$$= 0.$$