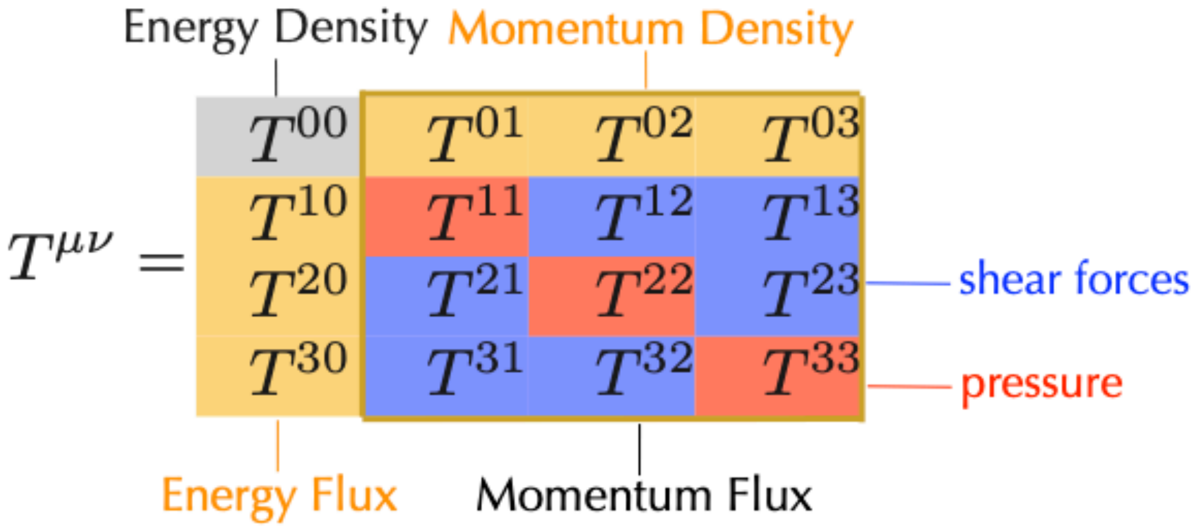
Energy-momentum tensor and GPDS

The Energy-Momentum Tensor



$$\langle N(p')|T^{\mu\nu}_{q,g}|N(p)\rangle$$

- Where does the spin of the proton come from?
- What are the mechanical properties (pressure, shear forces) inside the proton ?
- What is the origin of the proton mass?

Canonical Energy Momentum Tensor

Emmy Noether (1882-1935)

If a system has a continuous symmetry property, then there are corresponding quantities whose values are conserved in time

Translation invariance $\longrightarrow$ Conservation of the canonical EMT $T_C^{\mu\nu}(x)$

Lorentz invariance $\longrightarrow$ Conservation of the generalized Angular Momentum (AM) density

$$J_C^{\mu\alpha\beta}(x) = L_C^{\mu\alpha\beta} + S_C^{\mu\alpha\beta} \qquad L_C^{\mu\alpha\beta}(x) = x^\alpha T_C^{\mu\beta}(x) - x^\beta T_C^{\mu\alpha}(x)$$

Space components: $J_C^i(x) = \frac{1}{2}\epsilon^{ijk} J_C^{0jk}(x)$

$$\vec{J}_C = \vec{L}_C + \vec{S}_C$$

$\vec{L}_C$ → Orbital AM, $\vec{S}_C$ → Spin

$T_C^{\mu\nu}$ is in general neither gauge-invariant nor symmetric

$$T_C^{\mu\nu} - T_C^{\nu\mu} = -\partial_\lambda S_C^{\lambda\mu\nu}$$

Belinfante improved EMT

$$T_{\text{Bel}}^{\mu\nu}(x) = T_C^{\mu\nu}(x) + \partial_\lambda G^{\lambda\mu\nu}(x) \qquad \text{with } G^{\lambda\mu\nu} = -G^{\mu\lambda\nu}$$

Belinfante generalized AM

$$J_{\text{Bel}}^{\mu\alpha\beta}(x) = J_C^{\mu\alpha\beta}(x) + \partial_\lambda[x^\alpha G^{\lambda\mu\beta}(x) - x^\beta G^{\lambda\mu\alpha}(x)]$$

with the super-potential

$$G^{\lambda\mu\nu}(x) = \frac{1}{2}[S_C^{\lambda\mu\nu}(x) - S_C^{\mu\nu\lambda}(x) - S_C^{\nu\mu\lambda}(x)] = -G^{\mu\lambda\nu}(x)$$

↓

$$J_{\text{Bel}}^{\mu\alpha\beta}(x) = x^\alpha T_{\text{Bel}}^{\mu\beta}(x) - x^\beta T_{\text{Bel}}^{\mu\alpha}(x)$$

Belinfante, Rosenfeld (1940)

Canonical ⟷ Belinfante

in general not symmetric

$$T_C^{[\mu\nu]}(x) = -\partial_\alpha S^{\alpha\mu\nu}(x) \neq 0$$

$$[\mu\nu] = \mu\nu - \nu\mu$$

symmetric

$$T_{\text{Bel}}^{[\mu\nu]}(x) = 0$$

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clear distinction between OAM and spin at the density level

$$J_C^{\mu\alpha\beta}(x) = L_C^{\mu\alpha\beta}(x) + S_C^{\mu\alpha\beta}(x)$$

$$L_C^{\mu\alpha\beta}(x) = x^\alpha T_C^{\mu\beta}(x) - x^\beta T_C^{\mu\alpha}(x)$$

purely OAM density

$$J_{\text{Bel}}^{\mu\alpha\beta}(x) = x^\alpha T_{\text{Bel}}^{\mu\beta}(x) - x^\beta T_{\text{Bel}}^{\mu\alpha}(x)$$

Canonical ⟷ Belinfante

Canonical

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$$T_C^{[\mu\nu]}(x) = -\partial_\alpha S^{\alpha\mu\nu}(x) \neq 0$$

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Belinfante

symmetric

$$T_{\mathrm{Bel}}^{[\mu\nu]}(x) = 0$$

purely OAM density

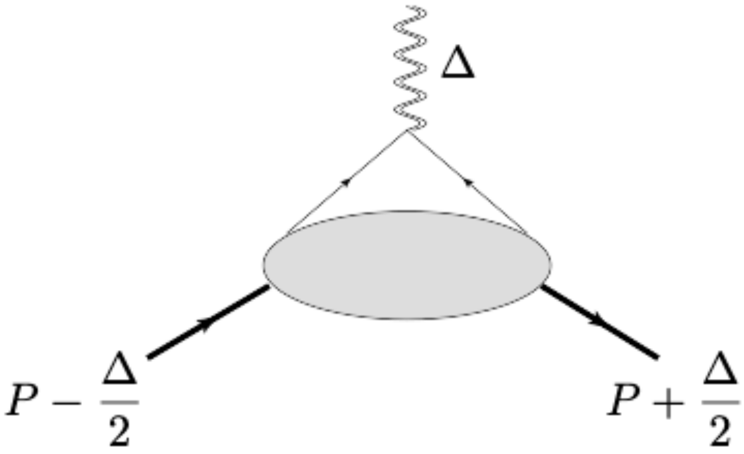
$$J_{\mathrm{Bel}}^{\mu\alpha\beta}(x) = x^\alpha T_{\mathrm{Bel}}^{\mu\beta}(x) - x^\beta T_{\mathrm{Bel}}^{\mu\alpha}(x)$$

The total charge does not change:

$$\int T_C^{0\nu}\, d^3x = \int T_{\mathrm{Bel}}^{0\nu}\, d^3x$$

$$\int J_C^{0\alpha\beta}\, d^3x = \int J_{\mathrm{Bel}}^{0\alpha\beta}\, d^3x$$

Form Factors of the EMT



Nucleon in external classical gravitational field

G couples to energy-momentum tensor
(symmetric-Belinfante)

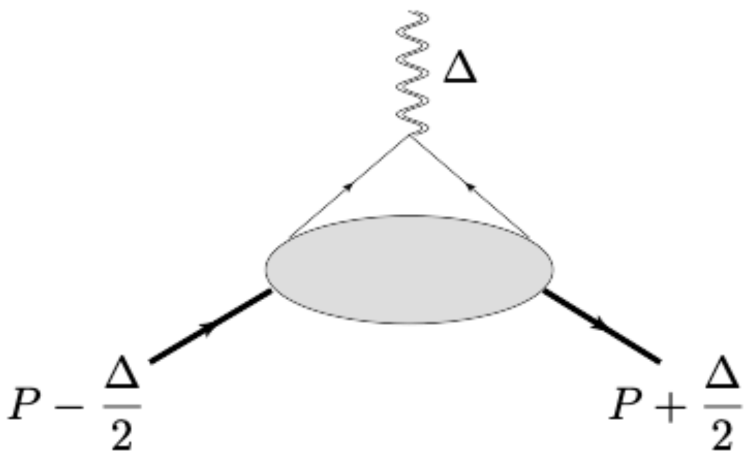
$$\langle P+\frac{\Delta}{2}|T^{\mu\nu}|P-\frac{\Delta}{2}\rangle$$

$$= \bar{u}(P+\frac{\Delta}{2})\left\{A(t)\frac{\gamma^{\{\mu}P^{\nu\}}}{2} + B(t)P^{\{\mu}i\sigma^{\nu\}\alpha}\frac{\Delta_\alpha}{4M}\right.$$

$$\left. + D(t)(\Delta^\mu\Delta^\nu - \Delta^2 g^{\mu\nu})\frac{1}{4M}\right\}u(P-\frac{\Delta}{2})$$

$$\{\mu,\nu\} \equiv (\mu\nu + \nu\mu)$$

Form Factors of the EMT



Nucleon in external classical gravitational field

G couples to energy-momentum tensor (symmetric-Belinfante)

$$\langle P + \frac{\Delta}{2} | T^{\mu\nu} | P - \frac{\Delta}{2} \rangle$$

$$= \bar{u}(P + \frac{\Delta}{2}) \left\{ A(t) \frac{\gamma^{\{\mu} P^{\nu\}}}{2} + B(t) P^{\{\mu} i\sigma^{\nu\}\alpha} \frac{\Delta_\alpha}{4M} + D(t)(\Delta^\mu \Delta^\nu - \Delta^2 g^{\mu\nu}) \frac{1}{4M} \right\} u(P - \frac{\Delta}{2})$$

$$\{\mu, \nu\} \equiv (\mu\nu + \nu\mu)$$

Gordon identity

$$\bar{u}' \gamma^\mu u = \bar{u}' \left(\frac{P^\mu}{M} + i\sigma^{\mu\alpha} \frac{\Delta_\alpha}{2M} \right) u$$

$$= \bar{u}(P + \frac{\Delta}{2}) \left\{ A(t) P^\mu P^\nu / M + (A(t) + B(t)) P^{\{\mu} i\sigma^{\nu\}\alpha} \frac{\Delta_\alpha}{4M} + D(t)(\Delta^\mu \Delta^\nu - \Delta^2 g^{\mu\nu}) \frac{1}{4M} \right\} u(P - \frac{\Delta}{2})$$

Momentum sum rule

$$\langle P|\hat{P}^{\nu}|P\rangle \quad = \quad \langle P|\int d^3\vec{x}\, T^{0\nu}(x)|P\rangle$$

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$$= \lim_{\Delta\to 0}\int d^3\vec{x}\, e^{-i\vec{x}\cdot\vec{\Delta}}\langle P+\frac{\Delta}{2}|T^{0\nu}(0)|P-\frac{\Delta}{2}\rangle$$

Momentum sum rule

$$\begin{aligned}
\langle P|\hat{P}^{\nu}|P\rangle &= \langle P|\int d^3\vec{x}\, T^{0\nu}(x)|P\rangle \\
&= \lim_{\Delta\to 0}\langle P+\frac{\Delta}{2}|\int d^3\vec{x}\, T^{0\nu}(x)|P-\frac{\Delta}{2}\rangle \\
&= \lim_{\Delta\to 0}\int d^3\vec{x} e^{-i\vec{x}\cdot\vec{\Delta}}\langle P+\frac{\Delta}{2}|T^{0\nu}(0)|P-\frac{\Delta}{2}\rangle \\
&= \lim_{\Delta\to 0}(2\pi)^3\delta^3(\vec{\Delta})\langle P+\frac{\Delta}{2}|T^{0\nu}(0)|P-\frac{\Delta}{2}\rangle
\end{aligned}$$

Momentum sum rule

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$$= \lim_{\Delta\to 0}(2\pi)^3\delta^3(\vec{\Delta})\langle P+\frac{\Delta}{2}|T^{0\nu}(0)|P-\frac{\Delta}{2}\rangle$$

Recall that:

$$\langle p',s'|T^{\mu\nu}_{\mathrm{Bel},a}(0)|p,s\rangle$$
$$= \bar{u}(p',s')\left[\frac{P^{\{\mu}\gamma^{\nu\}}}{2}A_a(t) + \frac{P^{\{\mu}i\sigma^{\nu\}\Delta}}{4M}B_a(t) + \frac{\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2}{4M}D_a(t) + Mg^{\mu\nu}\bar{C}_a(t)\right]u(p,s)$$

Momentum sum rule

$$\langle P|\hat{P}^{\nu}|P\rangle = \langle P|\int d^3\vec{x}\, T^{0\nu}(x)|P\rangle$$

$$= \lim_{\Delta\to 0}\langle P+\frac{\Delta}{2}|\int d^3\vec{x}\, T^{0\nu}(x)|P-\frac{\Delta}{2}\rangle$$

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&= \lim_{\Delta\to 0}(2\pi)^3\delta^3(\vec{\Delta})\langle P+\frac{\Delta}{2}|T^{0\nu}(0)|P-\frac{\Delta}{2}\rangle \\
&= A(0)\, P^{\nu}(2P^0)(2\pi)^3\delta^3(0)
\end{aligned}
$$

Momentum sum rule

$$
\begin{aligned}
\langle P|\hat{P}^{\nu}|P\rangle &= \langle P|\int d^3\vec{x}\,T^{0\nu}(x)|P\rangle \\
&= \lim_{\Delta\to 0}\langle P+\frac{\Delta}{2}|\int d^3\vec{x}\,T^{0\nu}(x)|P-\frac{\Delta}{2}\rangle \\
&= \lim_{\Delta\to 0}\int d^3\vec{x}\,e^{-i\vec{x}\cdot\vec{\Delta}}\langle P+\frac{\Delta}{2}|\,T^{0\nu}(0)|P-\frac{\Delta}{2}\rangle \\
&= \lim_{\Delta\to 0}(2\pi)^3\delta^3(\vec{\Delta})\langle P+\frac{\Delta}{2}|\,T^{0\nu}(0)|P-\frac{\Delta}{2}\rangle \\
&= A(0)\,P^{\nu}(2P^0)(2\pi)^3\delta^3(0) \\
&= A(0)\,P^{\nu}\,\langle P|P\rangle
\end{aligned}
$$

Momentum sum rule

$$\langle P|\hat{P}^{\nu}|P\rangle = A(0)P^{\nu}\langle P|P\rangle$$

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Momentum sum rule

$$\langle P|\hat{P}^{\nu}|P\rangle = A(0)P^{\nu}\langle P|P\rangle \qquad \langle P|\hat{P}^{\nu}|P\rangle = P^{\nu}\langle P|P\rangle$$

$$\Downarrow$$

$$\boxed{A(0) = 1}$$

Momentum sum rule

$$\langle P|\hat{P}^{\nu}|P\rangle \quad = \quad A(0)P^{\nu}\langle P|P\rangle \qquad\qquad \langle P|\hat{P}^{\nu}|P\rangle \quad = \quad P^{\nu}\langle P|P\rangle$$

$$\Downarrow$$

$$\boxed{A(0) = 1}$$

- Polinomiality: $\int dx\, xH(x,\xi,t) = A(t) + D(t)\,\xi^2$
- Physical interpretation in terms of:

Quarks $\quad A^q(0) = \int dx\, xH^q(x,0,0)$

&

Gluons $\quad A^g(0) = \int dx\, xH^g(x,0,0)$

- Total system: Momentum conservation

$$\boxed{A^q(0) + A^g(0) = 1}$$

Angular Momentum sum rule

Consider nucleon in the rest frame: $P^\mu = (M,0,0,0)$ $S^\mu = (0,0,0,1)$

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Consider nucleon in the rest frame: $P^\mu = (M, 0, 0, 0) \;\; S^\mu = (0, 0, 0, 1)$

$$\langle P, \frac{1}{2} | \hat{J}^3 | P, \frac{1}{2} \rangle \quad = \quad J \, \langle P, \frac{1}{2} | P, \frac{1}{2} \rangle$$

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Recall that: $\hat{J}^i = \frac{1}{2} \epsilon_{ijk} \int d^3\vec{x} J^{0jk}(x)$

$$J^{0jk}(x) = x^i T^{0j} - x^j T^{0i}$$

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$$= \quad \epsilon_{ij3} \langle P, \frac{1}{2} | \int d^3\vec{x} \, x^i T^{0j}(x) | P, \frac{1}{2} \rangle$$

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$$J^{0jk}(x) = x^i T^{0j} - x^j T^{0i}$$

$$= \epsilon_{ij3}\langle P, \frac{1}{2}|\int d^3\vec{x}\,x^i T^{0j}(x)|P, \frac{1}{2}\rangle$$

$$= \epsilon_{ij3}\lim_{\Delta\to 0}\langle P+\frac{\Delta}{2}, \frac{1}{2}|\int d^3\vec{x}\,x^i T^{0j}(x)|P-\frac{\Delta}{2}, \frac{1}{2}\rangle$$

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$$=\; \epsilon_{ij3}\lim_{\Delta\to 0}\int d^3\vec{x}\, x^i e^{-i\vec{x}\cdot\vec{\Delta}}\langle P + \frac{\Delta}{2}, \frac{1}{2}|T^{0j}(0)|P - \frac{\Delta}{2}, \frac{1}{2}\rangle$$

Angular Momentum sum rule

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$$= \epsilon_{ij3}\lim_{\Delta\to 0}\left[i\frac{\partial}{\partial\Delta^i}(2\pi)^3\delta^3(\vec{\Delta})\right]\langle P+\frac{\Delta}{2}, \frac{1}{2}|T^{0j}(0)|P-\frac{\Delta}{2}, \frac{1}{2}\rangle$$

Angular Momentum sum rule

$$= \epsilon_{ij3} \lim_{\Delta\to 0} (2\pi)^3 \delta^3(\vec{\Delta}) \left(-i\frac{\partial}{\partial \Delta^i}\right) \Big\{ [A(t)+B(t)] \bar{u}(p', 1/2)\, P^{\{0} i\sigma^{j\}\alpha} \frac{\Delta_\alpha}{4M} u(p, \frac{1}{2})$$

+ terms independent of Δ

+ terms quadratic in Δ $\Big\}$

$$= \epsilon_{ij3} (2\pi)^3 \delta^3(0) [A(0)+B(0)] \left(-\frac{1}{4M}\right) \bar{u}(P, \frac{1}{2}) \{P^0 \sigma^{ji} + P^j \sigma^{0i}\} u(P, \frac{1}{2})$$

$P^0 \to M$, $P^j \to 0$ in rest frame

$$= (2\pi)^3 \delta^3(0) [A(0)+B(0)] \left(\frac{1}{2M}\right) M\, \bar{u}(P, \frac{1}{2})\, \sigma^{12}\, u(P, \frac{1}{2})$$

$\bar{u}(P, \frac{1}{2})\, \sigma^{12}\, u(P, \frac{1}{2})$: $2M$ in rest frame

$$= \frac{1}{2} [A(0)+B(0)] \langle P, \frac{1}{2} | P, \frac{1}{2} \rangle$$

Angular Momentum sum rule

$$\langle P, \frac{1}{2}|\hat{J}^3|P, \frac{1}{2}\rangle = J \langle P, \frac{1}{2}|P, \frac{1}{2}\rangle = \frac{1}{2}[A(0) + B(0)]\langle P, \frac{1}{2}|P, \frac{1}{2}\rangle$$

$$J = \frac{1}{2}[A(0) + B(0)]$$

Angular Momentum sum rule

$$\langle P, \frac{1}{2} | \hat{J}^3 | P, \frac{1}{2} \rangle = J \, \langle P, \frac{1}{2} | P, \frac{1}{2} \rangle = \frac{1}{2} [A(0) + B(0)] \langle P, \frac{1}{2} | P, \frac{1}{2} \rangle$$

$$J = \frac{1}{2} [A(0) + B(0)]$$

- Total system: Angular Momentum conservation

$$\boxed{A(0) + B(0) = 1}$$

Angular Momentum sum rule

$$\langle P, \frac{1}{2} | \hat{J}^3 | P, \frac{1}{2} \rangle \; = \; J \, \langle P, \frac{1}{2} | P, \frac{1}{2} \rangle \; = \; \frac{1}{2} [A(0) + B(0)] \langle P, \frac{1}{2} | P, \frac{1}{2} \rangle$$

$$J \; = \; \frac{1}{2} [A(0) + B(0)]$$

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$$\boxed{A(0) + B(0) \; = \; 1}$$

- Polinomiality: $\int dx \, x H(x, \xi, t) = A(t) + D(t) \, \xi^2$ $\qquad \int dx \, x E(x, \xi, t) = B(t) - D(t) \, \xi^2$

Angular Momentum sum rule

$$\langle P, \frac{1}{2}|\hat{J}^3|P, \frac{1}{2}\rangle \;=\; J\,\langle P, \frac{1}{2}|P, \frac{1}{2}\rangle \;=\; \frac{1}{2}[A(0)+B(0)]\langle P, \frac{1}{2}|P, \frac{1}{2}\rangle$$

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- Angular momentum sum rule (Ji's relation)

$$J^{q,g} = \frac{1}{2}[A^{q,g}(0) + B^{q,g}(0)] = \frac{1}{2}\int dx\, x\;(H^{q,g}(x,\xi,0) + E^{q,g}(x,\xi,0))$$

Angular Momentum sum rule

$$\langle P, \frac{1}{2}|\hat{J}^3|P, \frac{1}{2}\rangle \;=\; J\,\langle P, \frac{1}{2}|P, \frac{1}{2}\rangle \;=\; \frac{1}{2}[A(0) + B(0)]\langle P, \frac{1}{2}|P, \frac{1}{2}\rangle$$

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$$J^{q,g} = \frac{1}{2}[A^{q,g}(0) + B^{q,g}(0)] = \frac{1}{2}\int dx\, x\, \left(H^{q,g}(x,\xi,0) + E^{q,g}(x,\xi,0)\right)$$

- Momentum + Angular Momentum conservation $\longrightarrow$ Gravitomagnetic sum rule

$$\begin{cases} A(0) = 1 \\ A(0) + B(0) = 1 \end{cases} \longrightarrow \boxed{B(0) = 0}$$

Kinetic EMT in QCD

Ji, 1997

$$T^{\mu\nu}_{\rm kin}(x) = T^{\mu\nu}_{{\rm kin},q}(x) + T^{\mu\nu}_{{\rm kin},g}$$

- Quark contribution: $T^{\mu\nu}_{{\rm kin},q}(x) = \frac{1}{2}\bar{\psi}(x)\gamma^\mu i\overleftrightarrow{D}^\nu\psi(x)$ $\qquad (D^\mu = \partial^\mu + igA^\mu)$

$$\boxed{\frac{1}{2}T^{\{\mu\nu\}}_{{\rm kin},q}(x) = T^{\mu\nu}_{{\rm Bel},q}(x)}$$

$$\frac{1}{2}T^{[\mu\nu]}_{{\rm kin},q}(x) = -\partial_\lambda S^{\lambda\mu\nu}_q(x)$$

$$S^{\lambda\mu\nu}_q(x) = \frac{1}{2}\epsilon^{\lambda\mu\nu\alpha}\bar{\psi}(x)\gamma_\alpha\gamma_5\psi(x)$$

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$$\boxed{\frac{1}{2}T^{\{\mu\nu\}}_{\text{kin},q}(x) = T^{\mu\nu}_{\text{Bel},q}(x)}$$

$$\frac{1}{2}T^{[\mu\nu]}_{\text{kin},q}(x) = -\partial_\lambda S^{\lambda\mu\nu}_q(x)$$

$$S^{\lambda\mu\nu}_q(x) = \frac{1}{2}\epsilon^{\lambda\mu\nu\alpha}\bar{\psi}(x)\gamma_\alpha\gamma_5\psi(x)$$

- Gluon contribution: $T^{\mu\nu}_{\text{kin},g} = -2\,\text{Tr}[F^{\mu\lambda}(x)F^\nu_\lambda(x)] + \frac{1}{2}g^{\mu\nu}\,\text{Tr}[F^{\rho\sigma}(x)F_{\rho\sigma}(x)]$

$$\boxed{T^{\mu\nu}_{\text{kin},g}(x) = T^{\mu\nu}_{\text{Bel},g}(x)}$$

Kinetic EMT in QCD

Ji, 1997

$$T^{\mu\nu}_{\text{kin}}(x) = T^{\mu\nu}_{\text{kin},q}(x) + T^{\mu\nu}_{\text{kin},g}$$

- Quark contribution: $T^{\mu\nu}_{\text{kin},q}(x) = \frac{1}{2}\bar{\psi}(x)\gamma^\mu i\overleftrightarrow{D}^\nu\psi(x)$ $\quad (D^\mu = \partial^\mu + igA^\mu)$

$$\boxed{\frac{1}{2}T^{\{\mu\nu\}}_{\text{kin},q}(x) = T^{\mu\nu}_{\text{Bel},q}(x)}$$

$$\frac{1}{2}T^{[\mu\nu]}_{\text{kin},q}(x) = -\partial_\lambda S^{\lambda\mu\nu}_q(x)$$

$$S^{\lambda\mu\nu}_q(x) = \frac{1}{2}\epsilon^{\lambda\mu\nu\alpha}\bar{\psi}(x)\gamma_\alpha\gamma_5\psi(x)$$

- Gluon contribution: $T^{\mu\nu}_{\text{kin},g} = -2\,\text{Tr}[F^{\mu\lambda}(x)F^\nu_{\ \lambda}(x)] + \frac{1}{2}g^{\mu\nu}\,\text{Tr}[F^{\rho\sigma}(x)F_{\rho\sigma}(x)]$

$$\boxed{T^{\mu\nu}_{\text{kin},g}(x) = T^{\mu\nu}_{\text{Bel},g}(x)}$$

Kinetic generalized AM

$$J^{\mu\alpha\beta}_{\text{kin},q}(x) = L^{\mu\alpha\beta}_{\text{kin},q}(x) + S^{\mu\alpha\beta}_q(x)$$

$$J^{\mu\alpha\beta}_{\text{Bel},q}(x) = J^{\mu\alpha\beta}_{\text{kin},q}(x) + \frac{1}{2}\partial_\lambda[x^\alpha S^{\lambda\mu\beta}_q(x) - x^\beta S^{\lambda\mu\alpha}_q(x)]$$

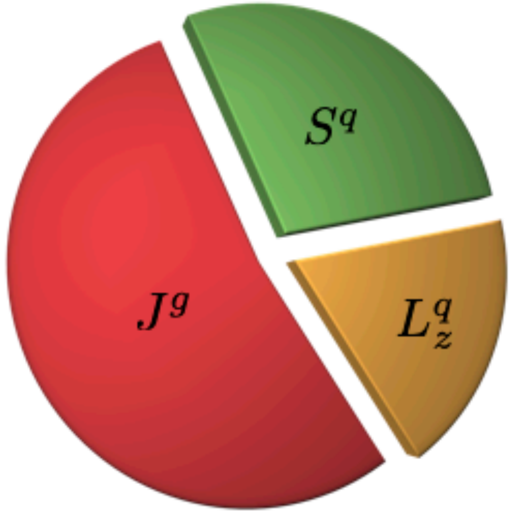
$$J^i = \frac{1}{2}\epsilon^{ijk}\int d^3x\,J^{0jk}$$

$$\int \vec{J}_{\text{Bel},q}\,d^3x = \int \vec{J}_{\text{kin},q}\,d^3x$$

equal total charge

Angular Momentum Relation (``Ji's Sum Rule)

*X. Ji, PRL **78** (1997) 610*



Ji's relation: $J^{q,g}(t=0) = \frac{1}{2}\int_{-1}^{1} dx\, x\, \left(H^{q,g}(x,\xi,0) + E^{q,g}(x,\xi,0)\right)$ $\quad (\xi = -\frac{\Delta^+}{2P^+})$

$H^{q,g}$: at $\xi = 0$ unpolarized PDF

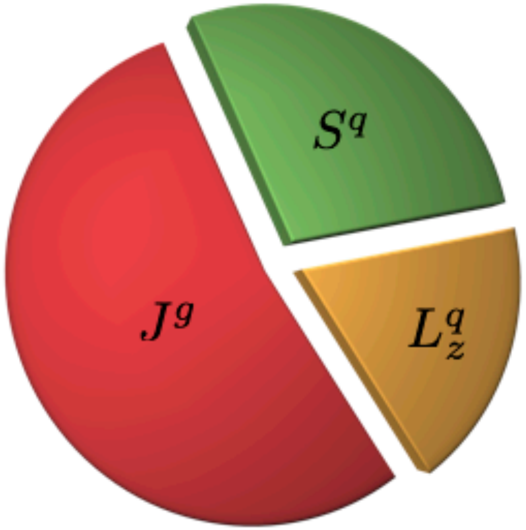
$E^{q,g}$: not directly accessible

$J^q = L^q + S^q \qquad S^q = \frac{1}{2}\int_0^1 dx\, g_1(x)$

J^g no further gauge-invariant decomposition

- Requires extrapolation at t=0
- Requires spanning x at fixed values of ξ ($\xi = 0$ is the most convenient)
- $J^q(x) \neq \frac{1}{2}x(H^q(x,0,0) + E^q(x,0,0)$ contribution from surface term

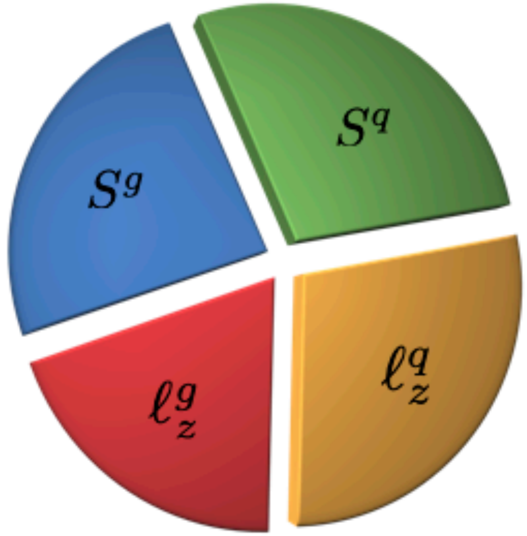
Ji (kinetic EMT) Sum Rule



$$\frac{1}{2} = \underbrace{S^q(\mu) + L_z^q(\mu)}_{J^q} + J^g(\mu)$$

- each term is gauge invariant
- frame independent
- it works also for the transverse AM in the infinite momentum frame
- J^q and J^g can be obtained from moments of GPDs

Jaffe-Manohar (canonical EMT) Sum Rule



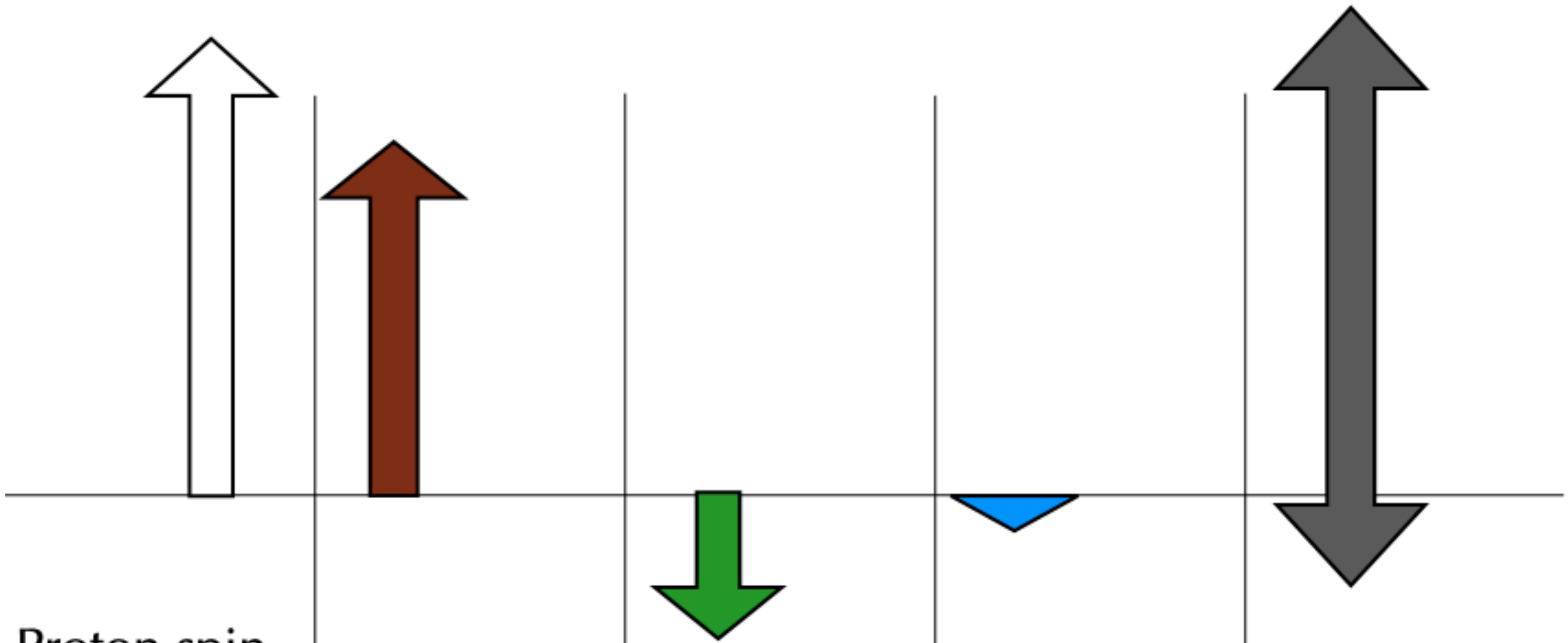
$$\frac{1}{2} = S^q(\mu) + \ell_z^q(\mu) + \ell_z^g(\mu) + S^g(\mu)$$

- ℓ_z^q, ℓ_z^g, S^g are gauge dependent, BUT measurable
- S^q, S^g can be obtained from pol. PDFs
- ℓ_z^q, ℓ_z^g can be obtained from twist-3 GPDs and Wigner distributions
- simple partonic interpretation in the IMF

$$\ell_z^q - l_z^{q,\mathrm{pot}} = L_z^q \longrightarrow \ell_z^g + S^g + l_z^{q,\mathrm{pot}} = J^g$$

Y. Guo, X. Ji, K. Shiells, NPB 96 (2021) 115440; X. Ji, F. Yuan, Y. Zhao, Nature Rev. Phys. 3 (2021) 1

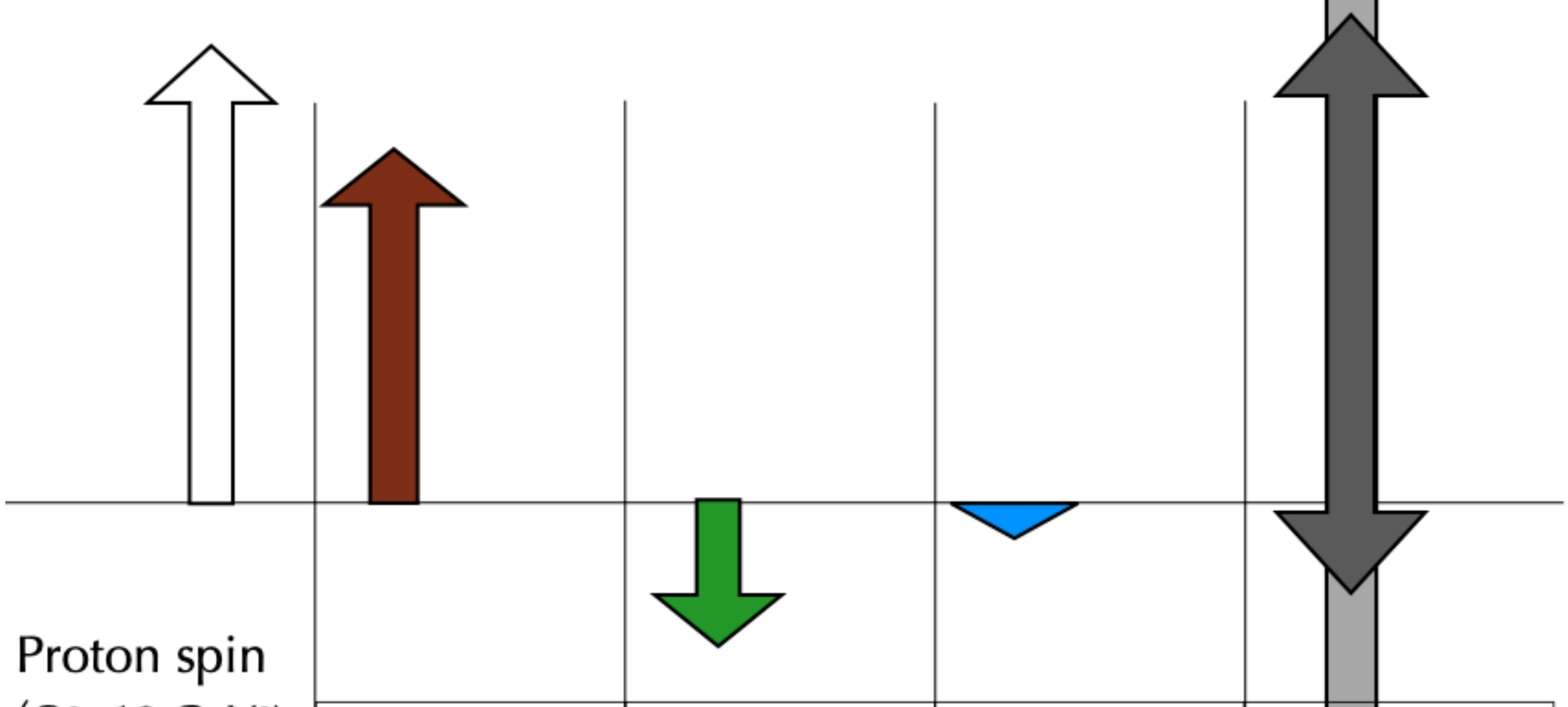
Status of Spin Sum Rule



Proton spin
(Q^2=10 GeV^2)

$u+\bar{u}$		$d+\bar{d}$		$s+\bar{s}$		g		
spin (%)	OAM	spin (%)	OAM	spin (%)	OAM	spin (%)	OAM	
81 ± 1		-43 ± 1		-11 ± 2		40 ± 60		DSSV14
79 ± 5		-45 ± 5		12 ± 8		3 ± 324		NNPDF

Status of Spin Sum Rule



Proton spin
(Q^2=10 GeV^2)

$u+\bar{u}$		$d+\bar{d}$		$s+\bar{s}$		g		
spin (%)	OAM	spin (%)	OAM	spin (%)	OAM	spin (%)	OAM	
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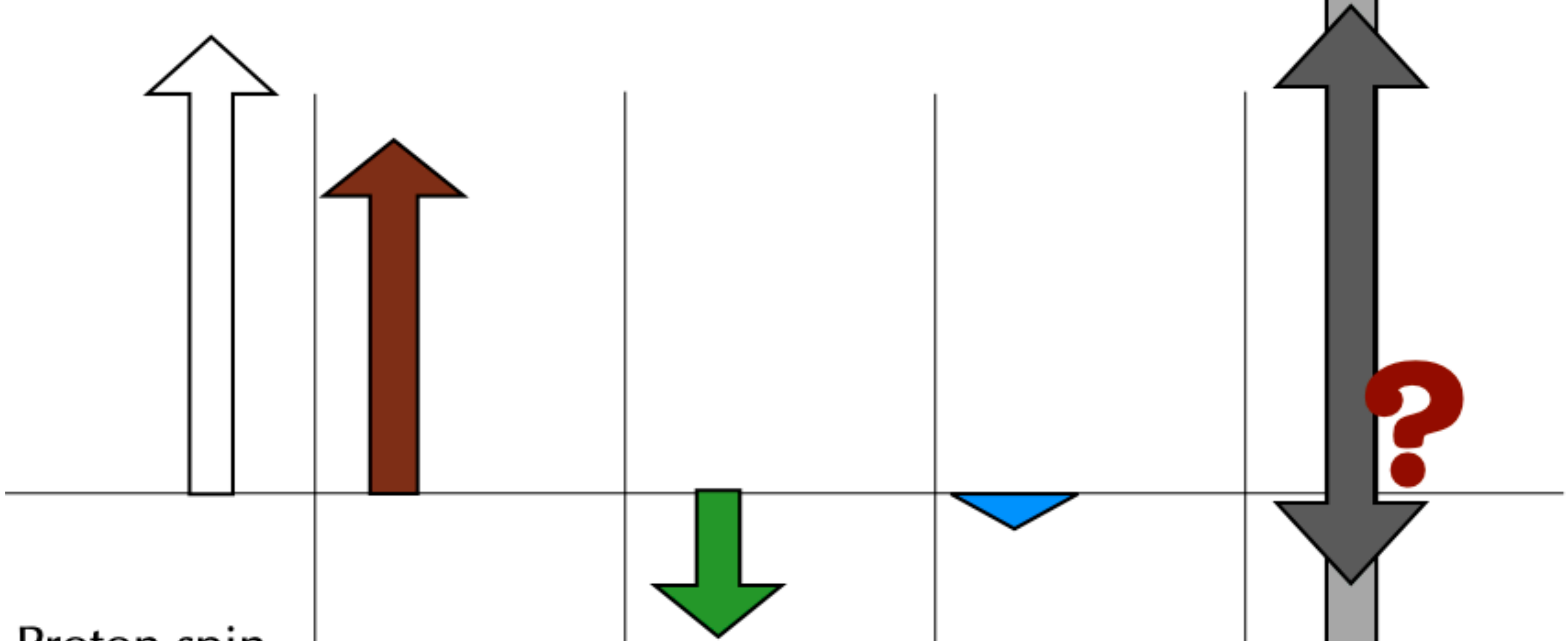

Proton spin
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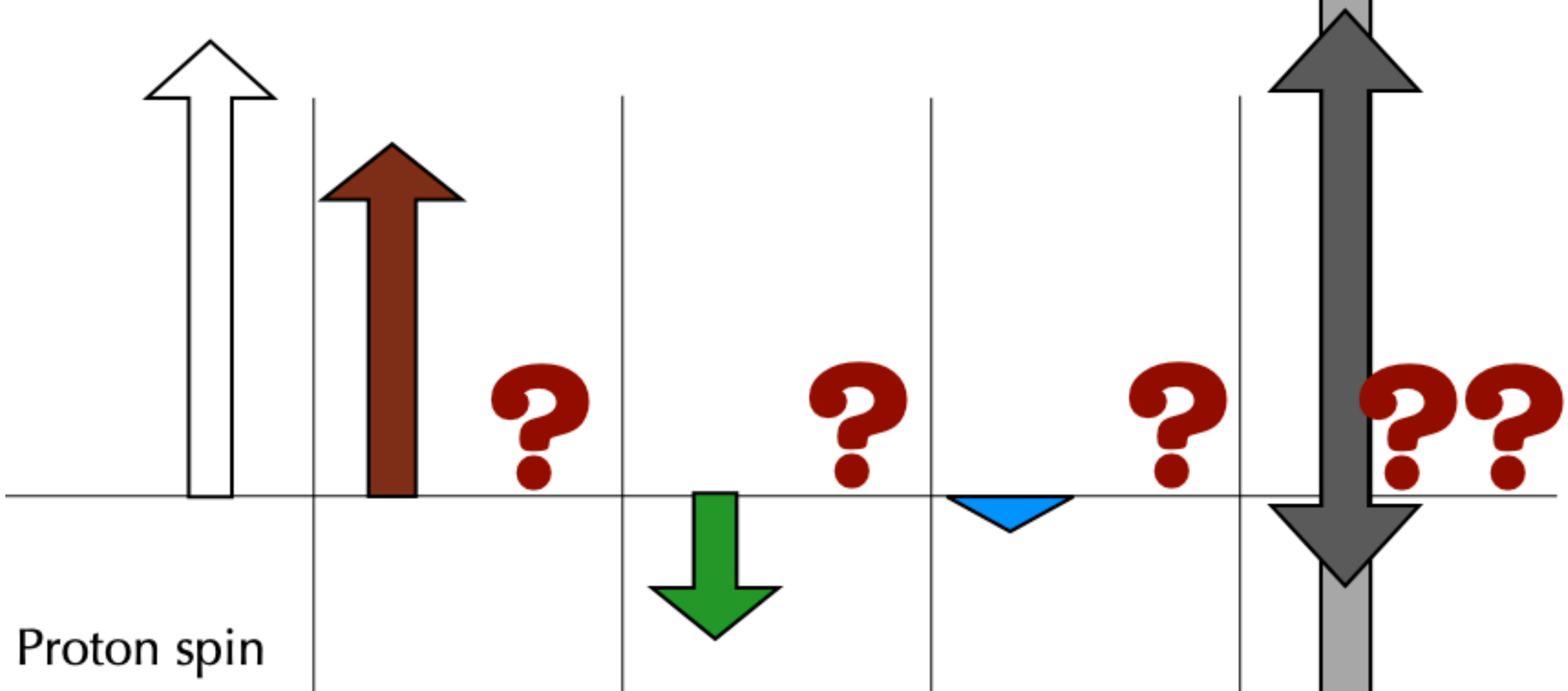
Status of Spin Sum Rule

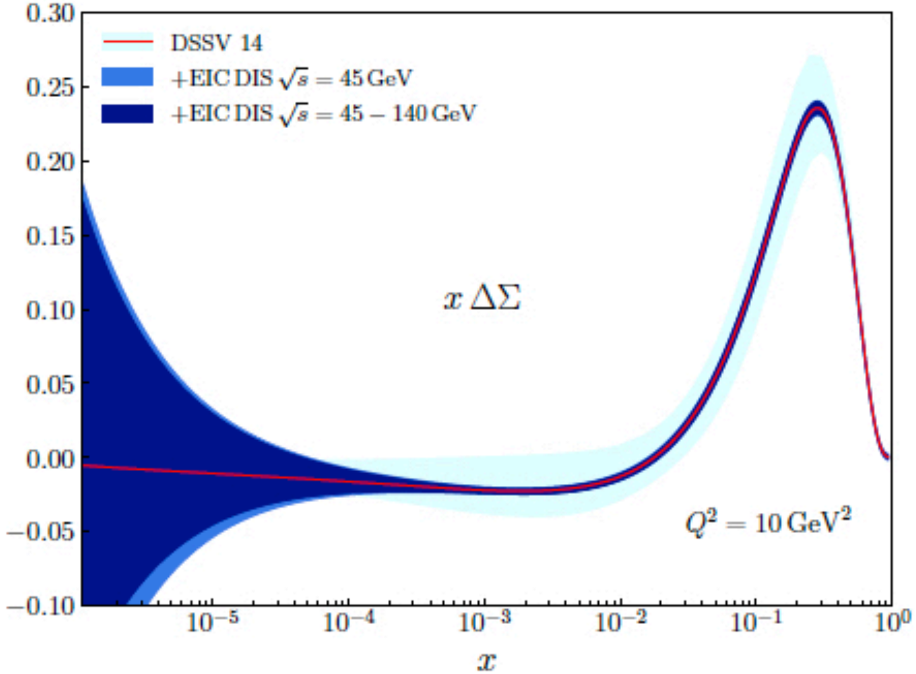


Proton spin
(Q^2=10 GeV^2)

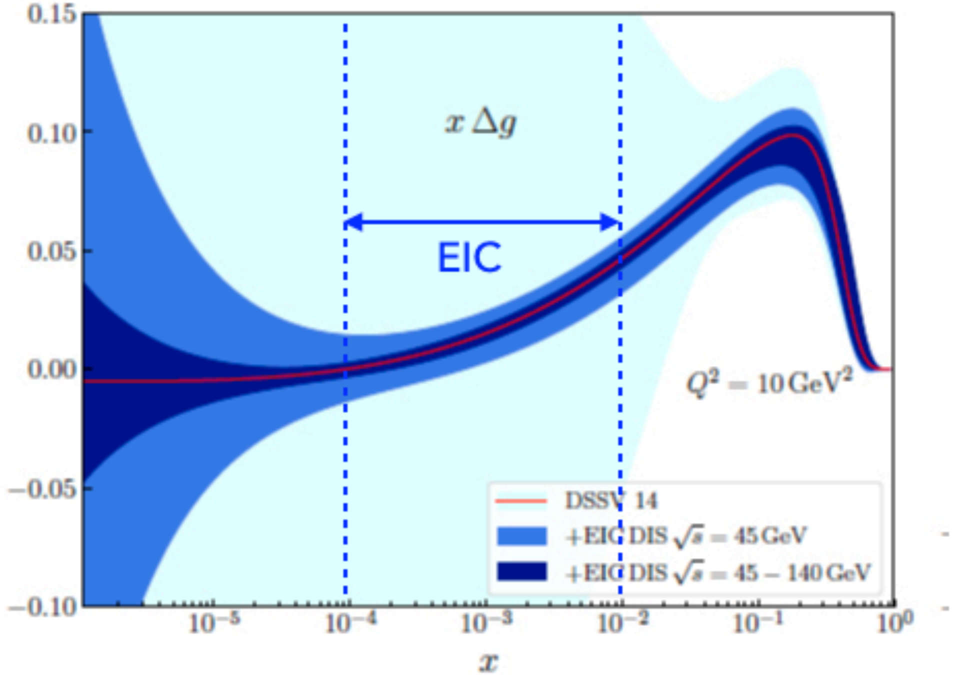
$u+\bar{u}$		$d+\bar{d}$		$s+\bar{s}$		g		
spin (%)	OAM	spin (%)	OAM	spin (%)	OAM	spin (%)	OAM	
81 ± 1		-43 ± 1		-11 ± 2		40 ± 60		DSSV14
79 ± 5		-45 ± 5		12 ± 8		3 ± 324		NNPDF

Impact of future EIC for quark and gluon spin contributions

Quark Spin



Gluon Spin



EIC Yellow Report: arXiv: 2103.05419

We are constantly improving the knowledge of the contributions to the spin of the nucleon

However the details on the flavor and sea contributions are still sketchy

What about a direct measurement of orbital angular momentum?

Orbital Angular momentum of the proton from available GPD measurements

$$J^{q,g} = \frac{1}{2}\int_{-1}^{1} dx\, x\, \left(H^{q,g}(x,\xi,0) + E^{q,g}(x,\xi,0)\right) \qquad L^q = J^q - S^q$$

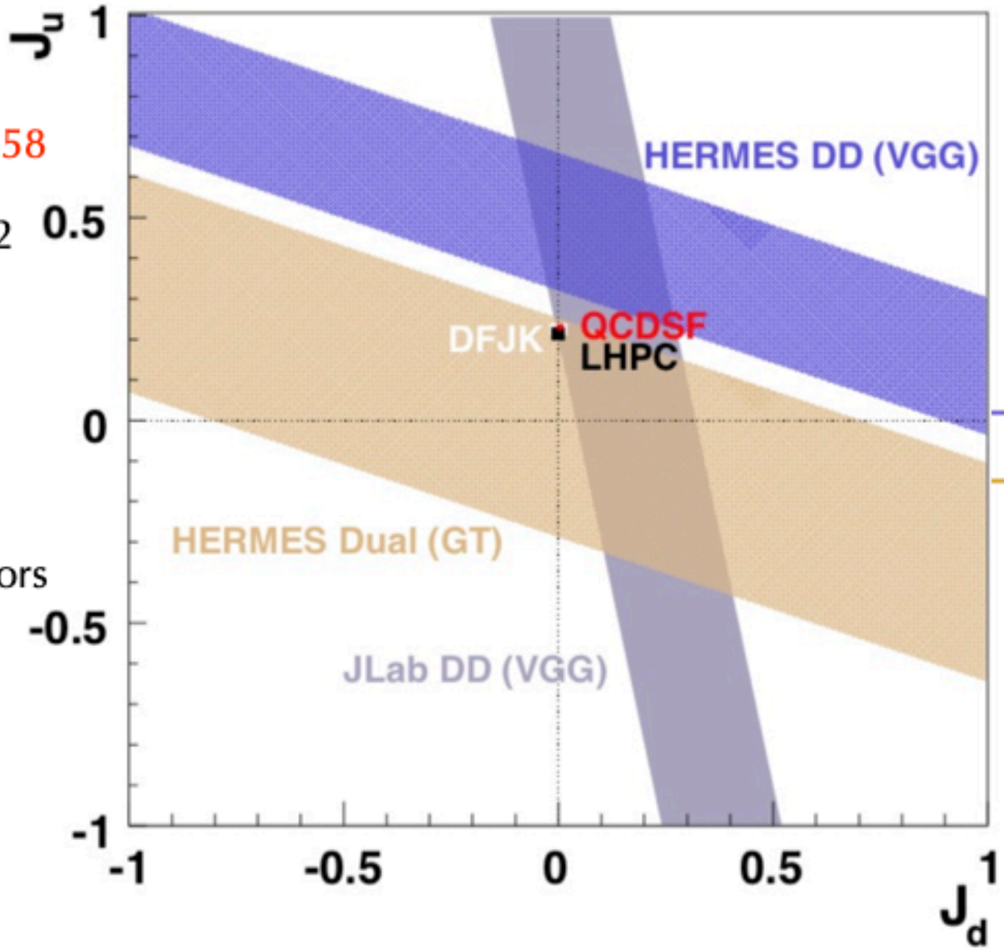
Lattice results

QCDSF: PoS (Lattice 2007) 158

LHPC: PRD77 (2008) 094502

GPDs extracted from form factors

DFJK, EPJC39 (2005) 1



extractions from HERMES data using two different models

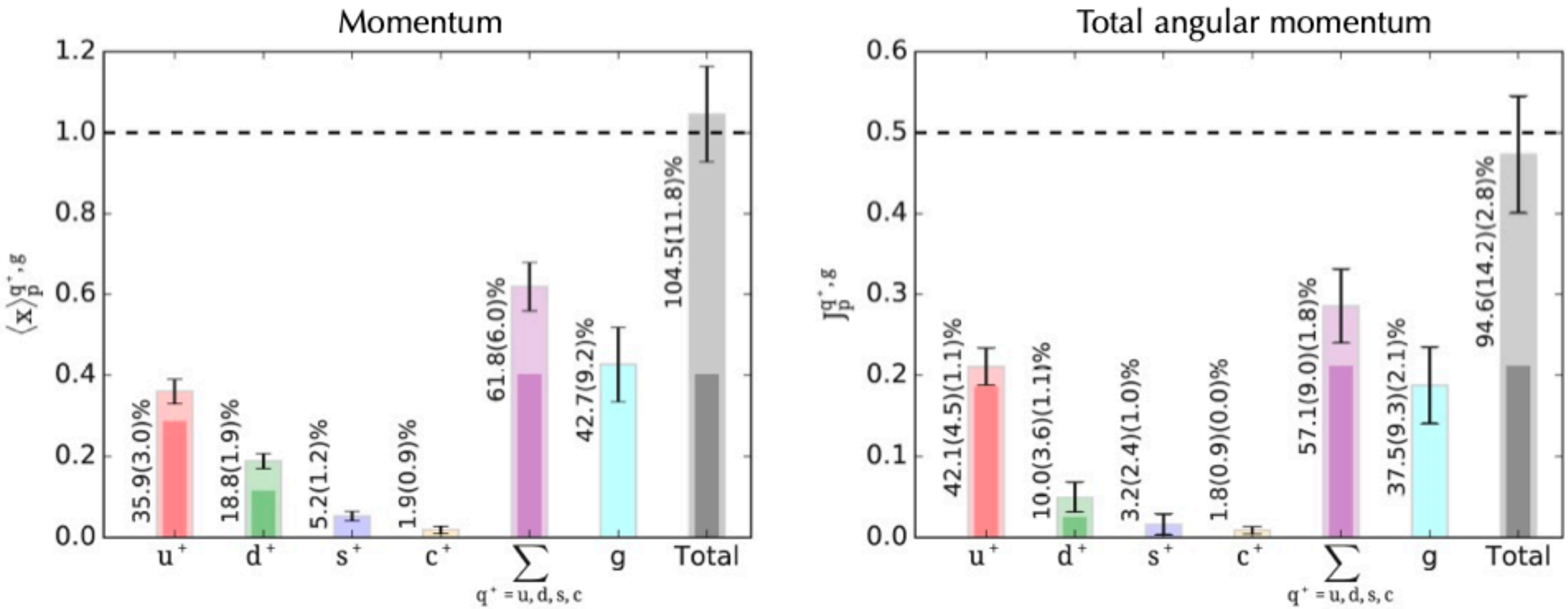
JLab Hall A, Phys. Rev. Lett. 99 (2007) 242501

Hermes Coll., JHEP 06 (2008) 066

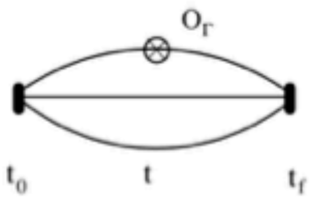
Improved accuracy with JLab12 and future EIC measurements!

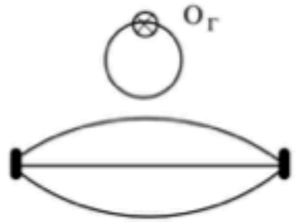
Sum rules from lattice QCD

- Results at physical pion mass at the scale of 2 GeV



- dark bars: connected
- light bars: disconnected contributions (quarks & gluons)



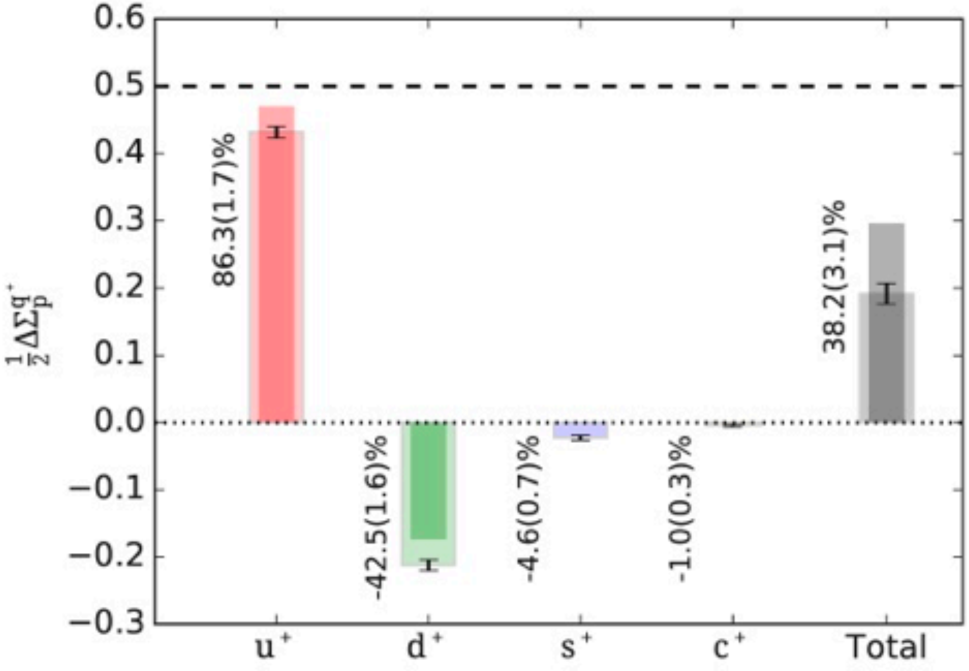


- gravitomagnetic sum rule: $\sum_{q=u,d,s} B^q(0) + B^g(0) = -0.099(91)(28)$

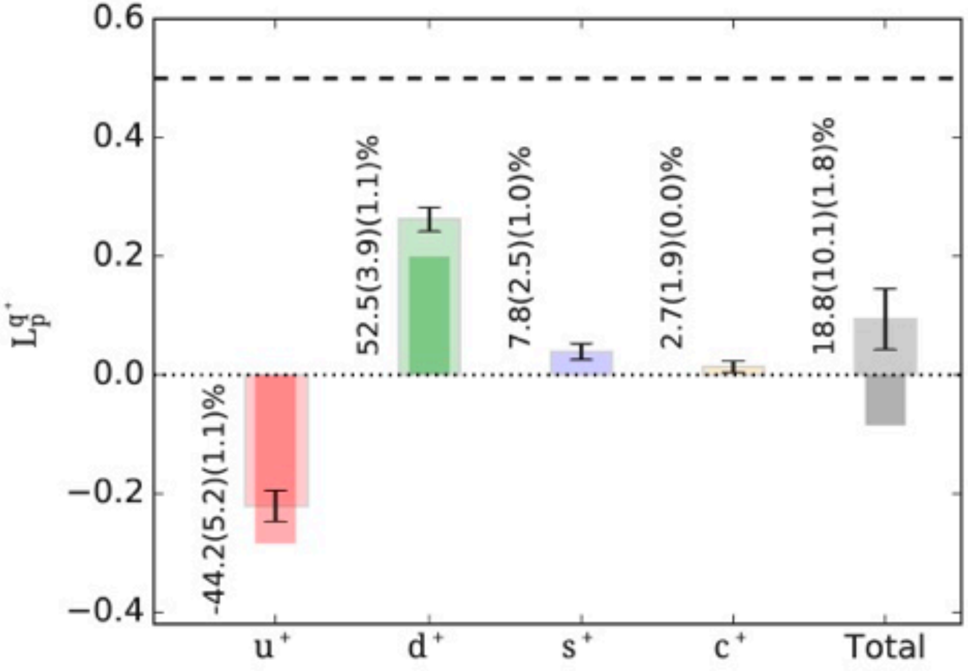
Alexandrou et al. [Extended Twist Mass Coll.], Phys.Rev.D 101 (2020), 094513

Spin and orbital angular momentum from lattice QCD

Quark spin



Quark orbital angular momentum $J^q - \frac{1}{2}\Delta\Sigma^q$



- dark bars: connected
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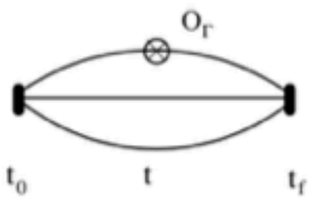
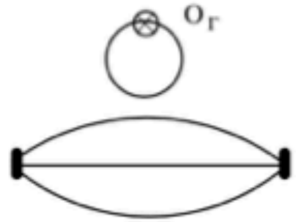
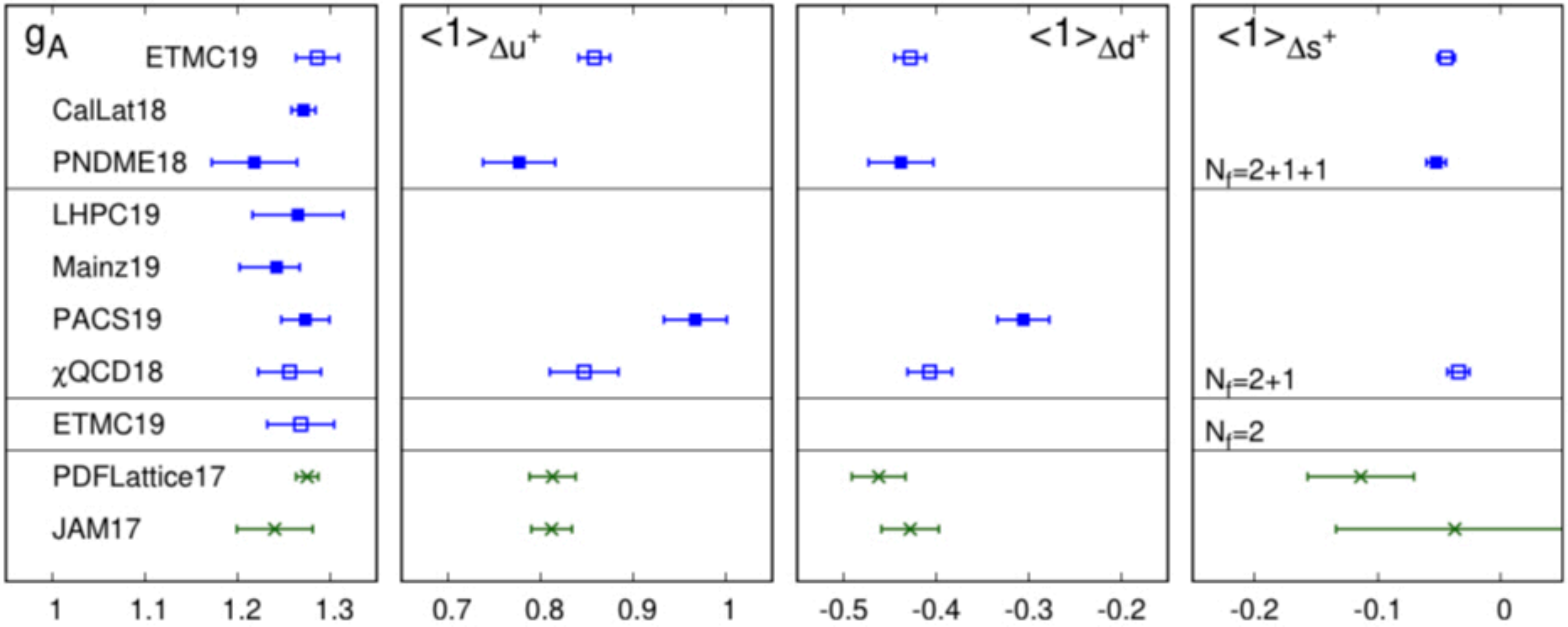




Alexandrou et al., Phys.Rev.D 101 (2020), 094513

Comparison Lattice QCD and Phenomenology

$g_A \equiv \langle 1 \rangle_{\Delta u^+} - \langle 1 \rangle_{\Delta d+}$ Quark spin $Q^2 = 4~\mathrm{GeV}^2$



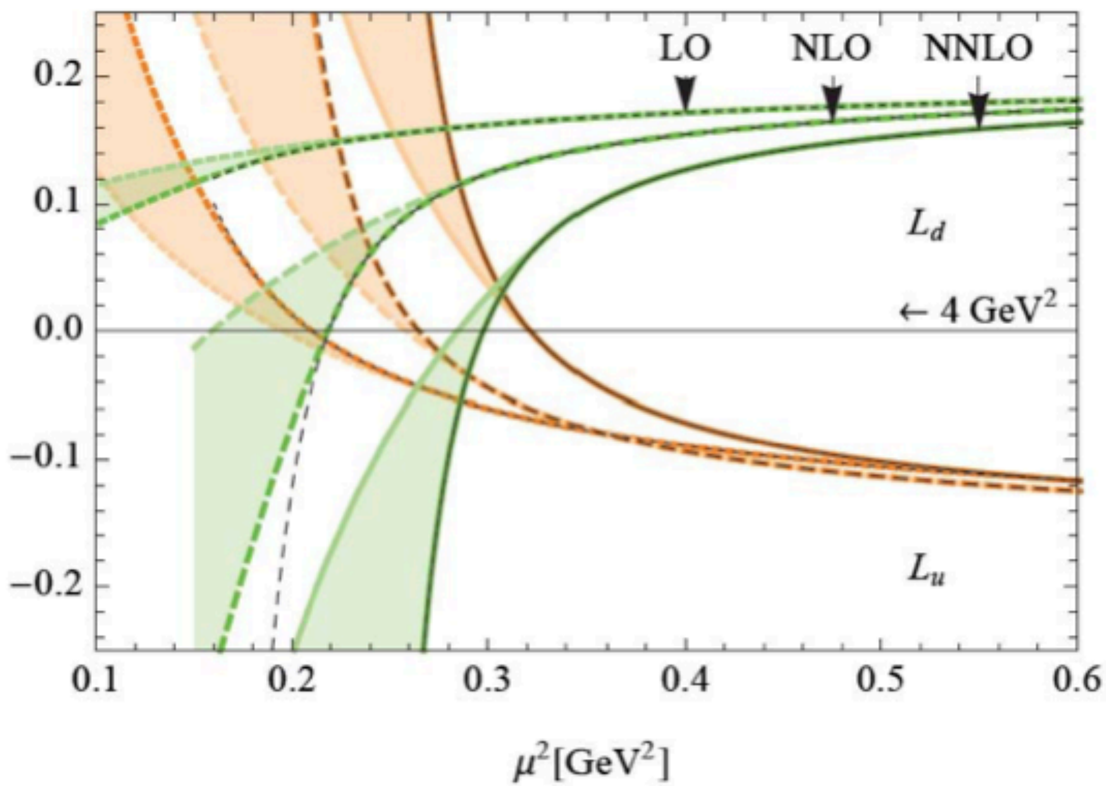
***PDFLattice17**: average of Lattice QCD results from the community white paper, Prog.Part.Nucl.Phys. 100 (2018) 107*

Overall fair agreement between lattice calculations and phenomenological fits

The uncertainties of the two have comparable size

Lattice QCD results could provide useful inputs to global fits of polarized PDFs

Scale dependence



Altenbuchinger, Hägler, Weise, EPJA47, 140 (2011)