Lectures on Non-Abelian FQH states

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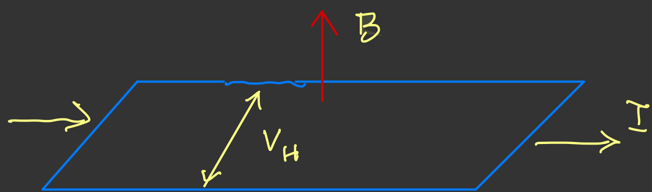
Outline

* Non-Abelian FQH states: why they are interesting, TQC, etc.
* Chern-Simons Gauge Theory and the bulk-boundary correspondence
* CFT representation of wave functions and the edge states
* Laughlin, Majorana, Ising, Parafermions and Fibonacci
* Physical mechanisms: pairing and clustering
* Field Theory of Non-Abelian FQH states: Moore-Read, Read-Rezayi, Fibonacci, Parafermions
* Outlook

Fractional Quantum Hall States

In the beginning .... two-dimensional electron gases in large magnetic field



$\sigma_{xy} = \nu \dfrac{e^2}{h}$, $\sigma_{xx} \to 0$ $(T \to 0)$

no dissipation

Laughlin: $\Psi_m(z_1, \ldots, z_N) \sim \prod_{i<j} (z_i - z_j)^m \; e^{-\frac{1}{4 l_0^2} \sum_{j=1}^{N} |z_j|^2}$ (1983)

filling fraction $\nu = \dfrac{1}{m}$; $\{z_j\}$: electron coordinates $(z = x + iy)$

$l_0 : (\hbar c / eB)^{1/2}$ magnetic length

Jain: composite fermion : electron + $(m-1)$ fluxes (m odd)

FQH state : IQH state of composite fermions

$\to \nu_{\pm}(p, s) = \dfrac{p}{2sp \pm 1}$ $\quad p = 1, 2, \ldots$ $\quad s = 0, 1, 2 \ldots$ (Laughlin : $p = 1$, $+$) $m = 2s + 1$

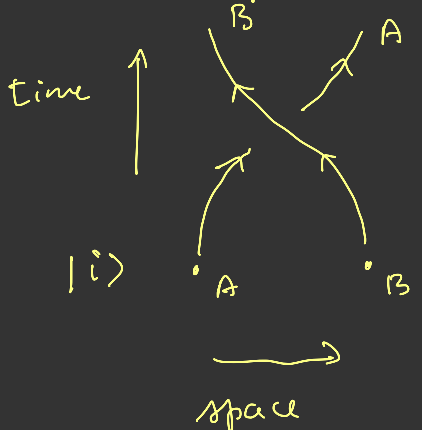
odd denominators

* The excitations of FQH fluids are vortices ("quasiholes") that

(a) carry fractional charge $q = \frac{1}{2sp \pm 1}$

(b) fractional braiding statistics

(c) m degenerate ground states on a torus (topological protection)



$$\psi'(A,B) = e^{i\varphi}\,\psi(B,A)$$

$\varphi = \frac{\pi}{2sp \pm 1} \Rightarrow$ anyons labelled by one-dimensional representations of the Braid Group

Hydrodynamic Derivation of the Effective Action

The three crucial physical properties of FQH's are

1. Incompressibility (i.e. a gap for all <u>bulk</u> excitations)

2. $\exists$ magnetic field $\Rightarrow$ broken time reversal (TR)

3. $\exists$ conserved (charge) current $j^\mu = (j_0, \vec{j})$ (Minkowski metric)

(3) $\Rightarrow$ $\partial_\mu j^\mu = 0 \iff j^\mu = \frac{e}{2\pi} \varepsilon^{\mu\nu\lambda} \partial_\nu a_\lambda$ $\leftarrow$ smooth

j^μ is invariant under $a_\mu \to a_\mu + \partial_\mu \varphi$ (gauge invariance)

what is the effective action for a_μ?

(1) $\Rightarrow$ locality, (2) $\Rightarrow$ odd under TR,

$$\mathcal{L} = \frac{m}{4\pi} \varepsilon_{\mu\nu\lambda} a^\mu \partial^\nu a^\lambda - \underbrace{\frac{e}{2\pi} \varepsilon_{\mu\nu\lambda} \partial^\nu a^\lambda}_{j_\mu} A^\mu - a_\mu \underset{\substack{\uparrow \\ \text{worldlines of the QP's}}}{j^\mu_{qp}}$$

Quantization of Chern-Simons Gauge Theory

$$\mathcal{L}_{CS} = \frac{k}{4\pi} \varepsilon_{\mu\nu\lambda} a^\mu \partial^\nu a^\lambda - j^\mu a_\mu \qquad (k \in \mathbb{Z})$$

If $j_\mu = 0$, $\mathcal{L}_{CS}$ is topological in the sense that it does not depend on the metric

j^μ is conserved $\iff$ $\partial_\mu j^\mu = 0$

In cartesian Coords.

$$\mathcal{L} = a_0 \left(\frac{k}{2\pi} \varepsilon_{ij} \partial_i a_j - j_0 \right) + \frac{k}{4\pi} \varepsilon_{ij} A^i \partial^0 A^j + \vec{J} \cdot \vec{a}$$

Lagrange multiplier $\Rightarrow$ $j_0 = \frac{k}{2\pi} b$; $b = \varepsilon_{ij} \partial_i a_j$

Gauss Law

① Eq. of motion: $\frac{k}{4\pi} \varepsilon_{\mu\nu\lambda} f^{\nu\lambda} = j_\mu \qquad f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$

If $j_\mu = 0 \Rightarrow f_{\mu\nu} = 0$ ("flat connections")

② a_1 and a_2 are canonical pairs $\quad [\hat{a}_1(\vec{x}), \hat{a}_2(\vec{y})] = i \frac{2\pi}{k} \delta^2(x-y)$

$$H = -\int d^2x \; \vec{j} \cdot \vec{a}$$

If $\vec{j} = 0 \Rightarrow H = 0$ no energy!

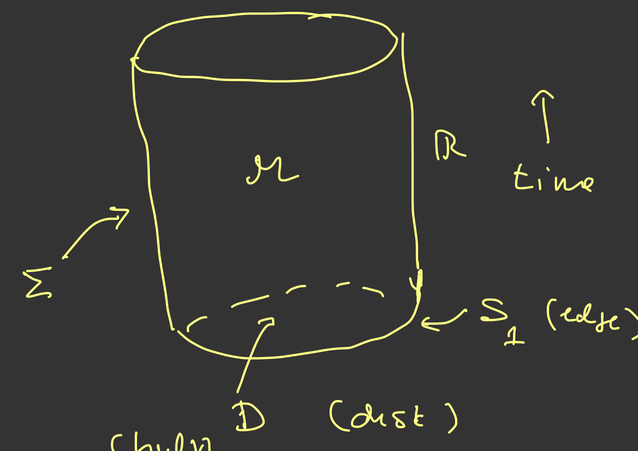
Suppose the space is a disk D with boundary $\partial D \cong S_1$ circle

The manifold is $\mathcal{M} = D \times \mathbb{R}$ ← time

Under gauge transf. , $a_\mu \to a_\mu + \partial_\mu \varphi$, the C.S. action changes

$$\delta S = \int_{\mathcal{M}} d^3x \; \partial^\mu (\Phi f^*_\mu) = \int_{S_1 \times \mathbb{R}} dS^\mu \; \varphi \, f^*_\mu \qquad f^*_\mu = \frac{1}{2} \varepsilon_{\mu\nu\lambda} f^{\nu\lambda}$$

$\Rightarrow \delta S = \varphi \times \text{flux}(\Sigma)$, $\Sigma = S_1 \times \mathbb{R}$



EOM $\Rightarrow$ $f_{\mu\nu} = 0$ $\Rightarrow$ $a_\mu = \partial_\mu \varphi$ (gauge transf.!)

$\Rightarrow$ the action integrates to the boundary $\Sigma = S_1 \times \mathbb{R}$ (cylinder)

S_1 ↑ edge, $\mathbb{R}$ ↑ time

$$\Rightarrow \quad S = \int_{S_1 \times \mathbb{R}} d^2x \, \frac{k}{4\pi} \partial_0 \varphi \, \partial_1 \varphi$$

$$\varphi(x_1 + L) = \varphi(x_1) \quad \text{(PBC's)}$$

We need to fix the gauge at the boundary $\mathcal{L}_{GF} = v \, a_1^2$

$$\Rightarrow \quad S_{eff}(\varphi) = \int_{S_1 \times \mathbb{R}} d^2x \, \frac{k}{4\pi} \left(\partial_0 \varphi \, \partial_1 \varphi - v (\partial_x \varphi)^2 \right) \quad \text{(real time!)}$$

* v here is arbitrary. In the context of the FQH states it is determined by the energetics at the edge. To leading order it is the drift velocity $\sim E/B$
* In CS theory it appears because <u>any</u> gauge fixing at the boundary breaks the large reparametrization invariance of the bulk theory

Observables of the edge theory

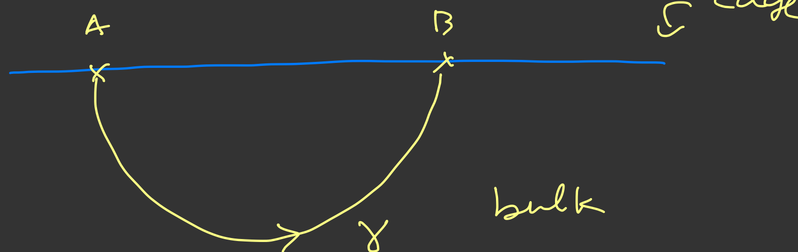
φ is not a physical field

$\frac{1}{2\pi}\partial_1 \varphi = j_1$ (edge current) $\Rightarrow \int_0^L dx_1\, j_1 = Q = \frac{(\Delta\varphi)_L}{2\pi}$

$Q = 0 \Leftrightarrow \Delta\varphi(L) = 0 \Leftrightarrow$ PBC's

$Q = r \Rightarrow \varphi(x_1+L) = \varphi(x_1) + 2\pi r$

$\Rightarrow e^-$ bulk $\Rightarrow r = m$



$$W_\gamma = e^{i\int_\gamma dx_\mu a^\mu}$$ Wilson arc

$$\equiv e^{i\int_\gamma dx_\mu \partial^\mu \varphi}$$

$$\equiv e^{i\varphi(B)}\, e^{-i\varphi(A)}$$

(path-ordered)

Observables at the edge are vertex operators: $V_p(x) = e^{ip\varphi(x)}$, p mod m

compactification radius $\varphi(x) \to \varphi(x) + 2\pi n R$ with $R = 1$

time-ordering

Propagators at the edge

$$\langle T\,\varphi(x,t)\;\varphi(0,0)\rangle = -\frac{1}{m}\ln\left(\frac{x-vt+i\varepsilon}{a_0}\right)$$

$x_0 = vt$

$x_1 = x$

electron operator $\qquad \nu = 1/m$

$$\Psi_e \equiv e^{im\varphi}$$

$$G_F(x,t) = \langle T\,\Psi_e^{\dagger}(x,t)\,\Psi_e(0,0)\rangle = \frac{\text{const}}{(x-vt+i\varepsilon)^m}$$

m fold pole!

$x \to -x$ and $t \to -t \Rightarrow G_F(x,t) = -G_F(-x,-t)$ if m is odd

$$V_1(\varphi) \equiv \Psi_{qp}$$

$$G_{qp}(x,t) = \langle T\,\Psi_{qp}^{\dagger}(x,t)\,\Psi_{qp}(0,0)\rangle = e^{\langle T\varphi(x,t)\varphi(0,0)\rangle} = \frac{\text{const.}}{(x-vt+i\varepsilon)^{1/m}}$$

branch cut!

$$\Rightarrow G_{qp}(x,t) = G_{qp}(-x,-t)\,e^{\pm i\frac{\pi}{m}}$$

fractional statistics!