L3 10/3/2023

CFT representation of FQH wave functions

* All abelian states can be written as expectation values in a 2D CFT

e.g. Laughlin $\Psi_m \sim \left\langle \prod_{i=1}^{N} e^{i\sqrt{m}\,\varphi(z_i)} \; e^{-\int d^2z' \sqrt{m}\, \rho_0 \varphi(z')} \right\rangle$

areal charge density

where $\varphi(z)$ is a compactified chiral boson in Euclidean space (imag. time)

$\varphi \sim \varphi + 2\pi\sqrt{m}$ (here I rescaled φ by $\sqrt{m}$ and $R = \sqrt{m}$)

* electron operator $V_e \sim e^{i\sqrt{m}\,\varphi(z)}$ scaling dimension $\Delta_e = \frac{m}{2}$

* quasi-hole (vortex) $V_{qh} \sim e^{i\varphi(z)/\sqrt{m}}$ scaling dimension $\Delta_{qp} = \frac{1}{2m}$

* physical states must be local w.r.t. the electron

$\Rightarrow$ only V_{qh}^{n} $(n = 1, \dots, m)$ are allowed

$\Rightarrow$ The ideal bulk wave functions are correlators of an Euclidean CFT! This is the same CFT of the edge but in imaginary time.

Non-Abelian States: Moore-Read (1991)

$$\Psi_{MR}(\{z_i\}) \sim \underbrace{\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right)}_{\text{Pfaffian}} \prod_{i<j} (z_i - z_j)^n \, e^{-\frac{1}{4\ell_0^2}\sum_{i=1}^N |z_i|^2}$$

Pfaffian: expectation value of chiral Majorana fermions $\quad \chi(z) = \chi^\dagger(z)$

Propagator: $\langle \chi(z)\, \chi(w) \rangle = \dfrac{1}{z-w}$

$$\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) = \langle \chi(z_1) \ldots \chi(z_N) \rangle$$

$\varphi(z)$: chiral boson $\quad \varphi(z) \sim \varphi(z) + 2\pi\sqrt{n} \quad ; \quad R = \sqrt{n}$

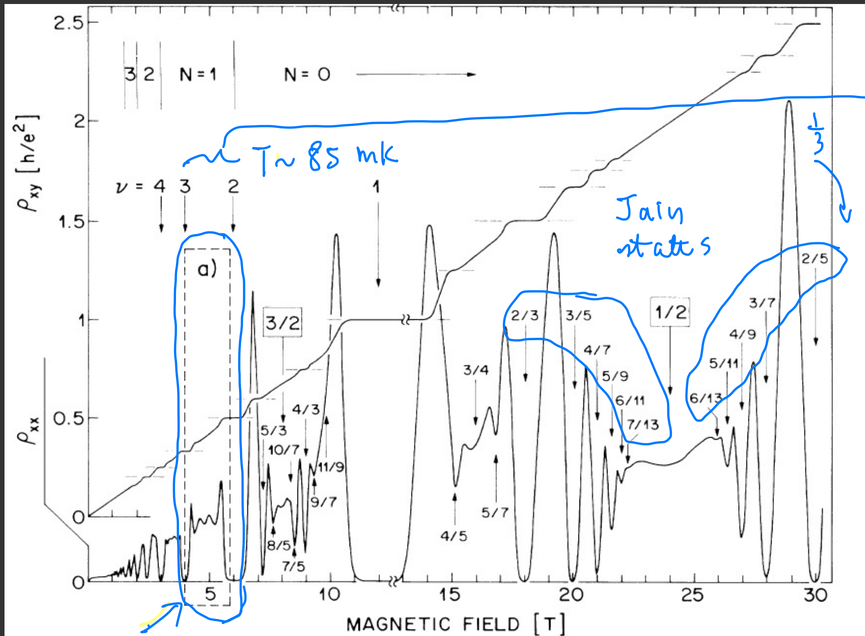
$$\Psi_{MR} \sim \langle \chi(z_1) \ldots \chi(z_N) \rangle \left\langle \prod_{i=1}^N e^{i\sqrt{n}\,\varphi(z_i)} \, e^{-\int d^2z' \sqrt{n}\, \rho_0\, \varphi(z')} \right\rangle$$

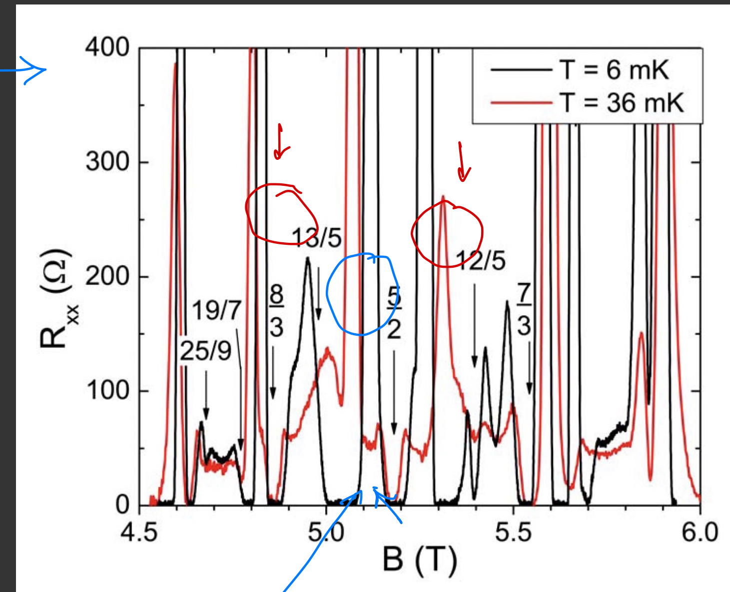
Filling fraction: $\nu = \dfrac{1}{n}$

n even $\rightarrow$ fermions ; n odd $\leftrightarrow$ bosons ; e.g. $\nu = \frac{1}{2}$ fermions

$\nu = 1$ bosons

Zoo of FQH states





Willett and Eisenstein ~ 1987

even denominator!
Moore-Read state?
W. Pan et al 2008

Fibonacci?

Most observed states are abelian (Jain)

First observation of $\frac{5}{2}$ state (even denominator)

$\left(\frac{5}{2} = 2 + \frac{1}{2}\right)$

Edge structure of the MR states

The structure of the wave function implies that the edge is a CFT with a charged mode and an edge mode.

Central charge

Charged mode: $\mathcal{L} = \frac{1}{4\pi}\left(\partial_0 \varphi\, \partial_1 \varphi - v_c (\partial_1 \varphi)^2\right)$ $\quad c = 1$

compactification radius $R = \sqrt{n}$

$V_p = e^{ip\varphi/\sqrt{n}}$; scaling dimensions: $\Delta_p = \frac{p^2}{2n}$

Neutral mode: chiral Majorana fermion χ

$\mathcal{L} = \chi\, i\partial_0 \chi - v_n\, \chi\, i\partial_1 \chi,$ $\quad \Delta_\chi = 1/2 \quad c = \frac{1}{2}$

Comment: this edge theory is supersymmetric

$C_T = \frac{3}{2}$

measured by thermal conductivity

* What are the allowed observables?

Observables of the MR Edge states

Naively one would expect to be the tensor product. NO

Electron Operator: $\psi_e \sim \underset{\Delta = 1/2}{\chi} \;\; \underset{\Delta = \frac{n}{2}}{e^{i\sqrt{n}\varphi}}$ $\qquad \Delta_e = \frac{n+1}{2}$, $Q = e$

n even, n+1 is odd

Propagator $\langle \psi_e(z)\, \psi_e^{+}(z') \rangle \sim \frac{1}{(z-z')^n} \times \frac{1}{z-z'} = \frac{1}{(z-z')^{n+1}}$ ✓

$(z = x_1 + i x_2)$

* $V_p(z) = e^{ip\varphi(z)/\sqrt{n}}$ has dimension $\Delta = \frac{p^2}{2n}$, $Q = \frac{pe}{n}$; $p = 0, \dots, n-1$

* Majorana $\chi(z)$, $\langle \chi(z)\, \chi'(z') \rangle = \frac{1}{z-z'}$, $\Delta = \frac{1}{2}$

* This is known as the Ising $\times$ $U(1)_n$ chiral CFT

Q: Are these all the anyons of these states? A: NO!

The Ising anyon

The Ising CFT has another primary field, the Ising field σ

* σ has dimension $\frac{1}{16}$
* σ is a "twist field": changes χ from PBC's to APBC's
* The charge sector has a "half vortex" $V_{1/2}(z) \sim e^{i\varphi(z)/2\sqrt{n}}$

$V_{1/2}$ is doubly-valued and it is non-local w.r.t. ψ_e

but $\psi_{qp} \sim \sigma(z)\, e^{i\varphi(z)/2\sqrt{n}}$ is local w.r.t. ψ_e!

This is the Ising anyon

$$Q = \frac{e}{2n}, \quad \Delta = \frac{1}{16} + \frac{1}{8n}$$

The Ising Anyon has non-abelian statistics

* Back to basics: the 2D Ising Model
* At T_c the 2D Ising Model (and the 1D Quantum Ising model at λ_c) is at a critical point $\Rightarrow$ scale invariance
* In 2D scale invariance $\Rightarrow$ conformal invariance

factorization

$$\langle \sigma(z,\bar{z})\ \sigma(w,\bar{w}) \rangle \sim \frac{\#}{(z-w)^{\Delta}\ (\bar{z}-\bar{w})^{\bar{\Delta}}}$$

$$\Delta = \bar{\Delta} = \frac{1}{16}$$

For us we will only need the chiral (analytic) part

$$\langle \sigma(z)\ \sigma(w) \rangle \sim \frac{\#}{(z-w)^{1/16}}$$

This expression enters in the wave function of two fundamental quasi holes

For two quasiholes at v_1 and v_2

$$\Psi_{2\text{-}qh} \sim \left\langle \prod_{i=1}^{N} \chi(z_i)\, e^{i\sqrt{n}\,\varphi(z_i)} \times \prod_{i=1}^{2} \sigma(v_i)\, e^{\frac{i}{2\sqrt{n}}\varphi(v_i)} \times \text{exp. factor} \right\rangle \times \text{Pfaffian}$$

This requires changing the Pfaffian factor (Nayak & Wilczek)

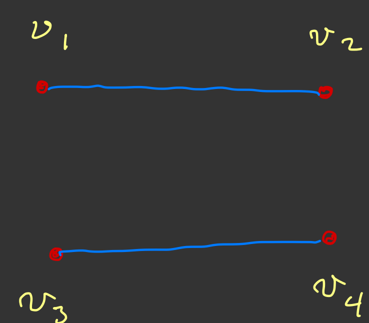
$$\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \rightarrow \mathrm{Pf}\left(\frac{(z_i - v_1)(z_j - v_2) + (z_j - v_1)(z_i - v_2)}{z_i - z_j}\right)$$

* How about four quasiholes at v_1, v_2, v_3, v_4 ?

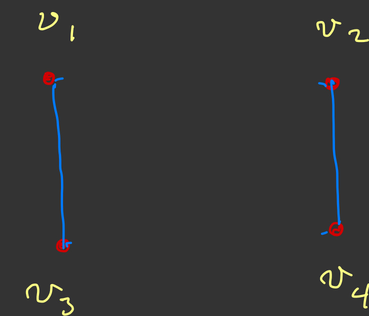
* There are two linearly independent wave functions, $|I\rangle$ and $|II\rangle$.

* For $2r$ qh's there are 2^{r-1} l.i. states with the same coords.!

* The pairings are branch cuts.



$|I\rangle$



$|II\rangle$

* Another way to understand this follows from the fusion rules of the Ising CFT

* Each pairing is the fusion of two spin fields σ

* (Chiral) Ising Fusion rules:

$$\sigma * \sigma = I + \chi \,, \quad \chi * \chi = I \,, \quad \sigma * \chi = \chi$$

* The first fusion rule $\Rightarrow$ two fusion channels: I, χ

$\Rightarrow$ The 4 qh wave function is not uniquely determined by the locations of the qh's but can be written as l.c. of two pairings

* Under a braiding (monodromy) these states transform under a unitary transf. $U = \frac{1}{\sqrt{2}} e^{i\pi/4} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \Rightarrow$ non-abelian fractional statistics!

Non-Abelian Fractional Statistics and Chern-Simons Theory

* To understand non-abelian fractional statistics we need a Chern-Simons theory with a non-abelian gauge group

* The gauge group is some non-abelian group G, such as $SU(2)$

* The vector potentials a_μ are matrices that can be written as linear combinations of the generators of $G = SU(N)$

For $SU(2)$ the matrices are the Pauli matrices $a_\mu(x) = a_\mu^I(x)\, \sigma_I$ Pauli

$$S = \frac{k}{8\pi} \int_{D\times\mathbb{R}} d^3x \left[\mathrm{tr}\left(\varepsilon_{\mu\nu\lambda} \left(a^\mu \partial^\nu a^\lambda \right) + \frac{2}{3} a^\mu a^\nu a^\lambda \right) \right] \qquad SU(2)_k$$

* This is the $SU(2)_k$ Chern-Simons theory
* The observables are Wilson loops with reps. $j = 0, \frac{1}{2}, \ldots, \frac{k}{2}$ $\quad (SU(2)_k)$
* Witten (1989) showed that the exp. value of a set of knotted Wilson loops is given by a topological invariant known as the Jones polynomial

If the geometry is a disk $D \times \mathbb{R}$

$\Rightarrow$ the edge theory is a chiral CFT in 2D known as the chiral (level k) Wess-Zumino-Witten theory

$$S_{WZW} = \frac{1}{4\lambda_c^2} \int_{S_1 \times \mathbb{R}} d^2x \, \mathrm{tr}(\partial_\mu g \, \partial^\mu g^{-1}) + \frac{k}{24\pi} \int_B d^3x \, \varepsilon_{\mu\nu\lambda} \mathrm{tr}(g^{-1}\partial^\mu g \, g^{-1}\partial^\nu g \, g^{-1}\partial^\lambda g)$$

$\lambda_c^2 = \frac{4\pi}{k}$, $g \in SU(N)$

This CFT was solved by Knizhnik & Zamolodchikov (1985)

It has a finite # of primaries ϕ_k. The fusion rules are:

For $SU(2)_k$ $\quad \phi_{j_1} * \phi_{j_2} = \phi_{|j_1 - j_2|} + \cdots + \phi_{k/2}$

* $SU(2)_1$ $\quad j = 0, \frac{1}{2}$ (only) $\quad [\frac{1}{2}] * [\frac{1}{2}] = [0]$
* $SU(2)_2$ $\quad j = 0, \frac{1}{2}, 1$ and $\quad \underbrace{[\tfrac{1}{2}] * [\tfrac{1}{2}] = [0] + [1] \, ; \, [1] * [1] = [0]}_{\text{Ising fusion rules}}$

etc.

To understand why this is the right theory we will look at two examples: bosons at $\nu = \frac{1}{2}$ and 1

① Bosons at $\nu = 1/2$

* We saw that one description was in terms of $U(1)_2$ Chern-Simons
* It has two anyons: I, $V_{1/2} = e^{i\varphi/\sqrt{2}}$ and the boson is $\psi_b \sim e^{i\sqrt{2}\varphi}$
* At the edge, the boson $e^{\pm i\sqrt{2}\varphi}$ have scaling dimension $\Delta_{1/2} = 1$
* The current $\sim \partial_x \varphi$ also has $\Delta = 1$
* It can be shown that they form an algebra $J_3 \sim \partial\varphi$, $J_{\pm} \sim e^{\pm i\sqrt{2}\varphi}$

and that J_3, $J_{\pm}$ satisfy the algebra of $SU(2)$ with $k=1$

$\Rightarrow$ This edge is that of $SU(2)_1$ WZW theory

* $U(1)_2$ is dual to $SU(2)_1$

(2) The case of MR for bosons at $\nu = 1$

$$\Psi^{\nu=1}_{MR} \sim \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i<j} (z_i - z_j)^1 \times \text{exp. factors}$$

(Chiral) Majorana χ $\qquad$ φ with $R = \sqrt{2}$

* Three currents $J^3 \sim \partial_x \varphi$ and $J^\pm \sim \chi\, e^{\pm i\sqrt{2}\varphi}$ satisfy the algebra of $SU(2)_2$

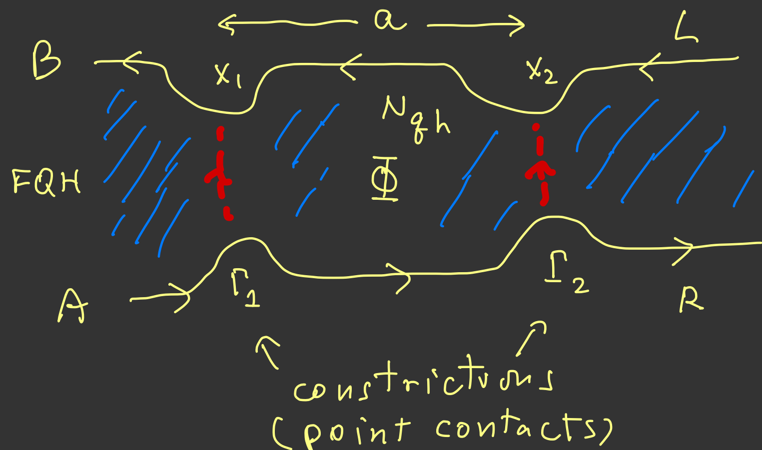
	I	$\sigma e^{i\varphi/2}$	$e^{\pm i\varphi}$	χ	$\chi e^{\pm i\varphi}$
(j, m)	$(0,0)$	$(\frac{1}{2}, \pm\frac{1}{2})$	$(1, \pm 1)$	$(1, 0)$	current
Δ	0	$3/16$	$1/2$	$1/2$	1
Q	0	$1/2$	0	1	0

$$\sigma e^{i\frac{\varphi}{2}} \times \sigma e^{-i\frac{\varphi}{2}} \sim I + \chi \qquad \text{Non-abelian !}$$

$\Rightarrow$ bulk theory should be CS $SU(2)_2$

FQH Quantum Interferometers (Chamon et al, 1997)

(Fabry - Perot)



$\nu = \frac{1}{m}$ Laughlin state

$$\Gamma_{1,2} \equiv \bar{\Gamma}_{1,2}\, e^{\pm i\pi\nu\Phi/\phi_0} \qquad \phi_0 = \frac{hc}{e}$$

voltage

$$\omega_0^* = e^* V/\hbar$$

$$\mathcal{L} = \frac{1}{8\pi}\left((\partial_0\varphi)^2 - v^2(\partial_1\varphi)^2\right) - \sum_{j=1,2} \Gamma_j\, e^{-i\omega_0^* x_0}\, \delta(x-x_j)\, e^{i\sqrt{\nu}\,\varphi(t,x_j)} + h.c$$

Total tunneling current: $I = I_1 + I_2 = e^* |\Gamma_{eff}|^2 \dfrac{2\pi}{\Gamma(2\nu)} |\omega_0^*|^{2\nu-1} \operatorname{sgn}(\omega_0^*)$

$$|\Gamma_{eff}|^2 = |\Gamma_1|^2 + |\Gamma_2|^2 + (\Gamma_1\Gamma_2^* + \Gamma_1^*\Gamma_2)\, F_\nu\left(\frac{\omega_0^* a}{v}\right)$$

$$\Gamma_1^*\Gamma_2 = \bar{\Gamma}_1^*\bar{\Gamma}_2 \exp\left(-i\,2\pi\left(\nu\frac{\Phi}{\phi_0} - N_{qh}\,\nu\right)\right)$$

Aharonov-Bohm + fractional statistics

The non-abelian FQH Interferometer

(Fradkin, Nayak, Wilczek, Tsvelik, 1998)

(Bonderson, Kitaev, Shtengel, 2006; Stern, Halperin, 2006)

Same geometry but now we consider tunneling of Ising anyons

Ising anyon quasihole: $\sigma \exp\left(\frac{i}{2\sqrt{2}}\varphi\right)$ (MR @ $\nu = 1/2$)

* Inject a qh at A (lower edge) $\to$ tunnels @ $x_1 \longrightarrow$ arrives @ B as $|\psi\rangle$
* Inject a 2nd qh at A $\longrightarrow$ tunnels @ $x_2 \to$ arrives at B as $e^{i\alpha} B_{N_{qh}} |\psi\rangle$
* α: AB phase; $B_{N_{qh}}$ is the braiding operator for the 2nd qh to encircle the N_{qh} trapped inside the interferometer

$$\Rightarrow \sigma_{xx} \sim |\Gamma_1|^2 + |\Gamma_2|^2 + \mathrm{Re}\left(\Gamma_1^* \Gamma_2 e^{i\alpha} \langle\psi| B_{N_{qh}} |\psi\rangle\right)$$

$\langle\psi| B_{N_{qh}} |\psi\rangle$: exp. value of N_{qh} Wilson loop operators

$= V_{N_{qh}}(e^{i\pi/4})$: Jones Polynomial (Witten, 1989)

The $\nu = \frac{1}{2}$ MR state involves a deformation of $SU(2)_2$

Even-odd effect (Stern & Halperin; Bonderson, Kitaev & Shtengel)

N_{qh} odd $\Rightarrow$ $\sigma_{xx} \sim |\Gamma_1|^2 + |\Gamma_2|^2$ (i.e. $\langle \psi | B_{N_{qh}} | \psi \rangle = 0$)

N_{qh} even $\Rightarrow$ $\sigma_{xx} \sim |\Gamma_1|^2 + |\Gamma_2|^2 + 2|\Gamma_1||\Gamma_2| (-1)^{N_\psi} \cos\left(\alpha + \arg\left(\frac{\Gamma_2}{\Gamma_1}\right) + N_{qh}\frac{\pi}{4}\right)$

$N_\psi = 1$ if N_{qh} fuse into $|\psi\rangle$] follows from the fusion rules.

$N_\psi = 0$ otherwise

Problem: the Laughlin qp spoils the even-odd effect.

$\Rightarrow$ The linearly indep. states are topological qubits

$\Rightarrow$ Platform for TQFT (Freedman, Kitaev, Larsen, 2002; Kitaev 1997)

$\Rightarrow$ Not universal: need Fibonacci)

Pairing and non-abelian states

Q: what is the physical meaning of the Pfaffian factor?

* It means that electrons can be closer to each other than in a Laughlin state.
* This suggests that there is an effective attractive interaction
* $\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right)$ is the wave function of a $p_x + i p_y$ superconductor in the weak-pairing (BCS) regime (Read and Green)
* This state arises in the $N = 1$ Landau level due to the structure of the one-particle Landau states
* Pairing of composite fermions
* But, this is NOT a superconductor!
* The $p_x + i p_y$ SC has a half-vortex $\Longleftrightarrow$ Ising anyon
* This is an example of a cluster state

Generalization: Read-Rezayi states (RR) (1998)

Based on $\mathbb{Z}_k$ <u>parafermions</u> (and $SU(2)_k$)

$$\psi_n(z) * \psi_m(z') \sim \frac{1}{(z-z')^{\Delta_n+\Delta_m-\Delta_{n+m}}} \psi_{n,m}(z') + \dots$$

Fradkin & Kadanoff (1980) (!)

$$\Delta_n = \frac{n(k-n)}{k}, \quad n,m = 1,\dots,k-1$$

RR states use the Parafermion CFT (Zamolodchikov & Fateev, 1985 Gepner & Qiu, 1987)

$$\Psi_{RR}(\{z_i\}) \sim \langle \psi_1(z_1) \cdots \psi_1(z_N) \rangle \prod_{i<j} (z_i - z_j)^{M+\frac{2}{k}} \times \text{gaussians}$$

vanishes when $k+1$ particles come together. clustering

$M \in \mathbb{Z}$ divisible by k; M even: bosons, M odd: fermions; $\nu = \frac{k}{Mk+2}$

The most interesting case is $k=3$ ($\mathbb{Z}_3$) ($\nu = \frac{3}{2}$ (B), $\frac{3}{5}$ (F))

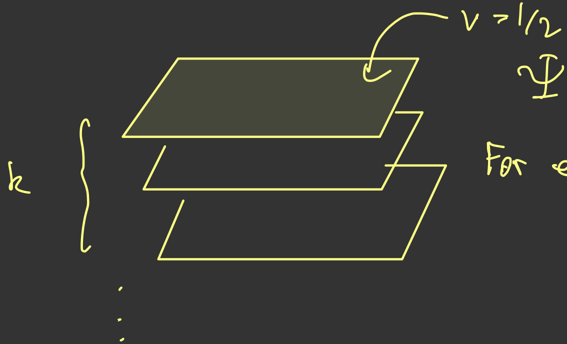
In addition to the $\mathbb{Z}_3$ parafermion, it has a Fibonacci anyon τ

Fusion rule: $\tau * \tau = I + \tau$ $\Rightarrow$ its unitary braiding matrices cover $SU(2)$
(Fibonacci sequence) $\Rightarrow$ <u>universal</u> quantum computer

Effective Field Theory Approaches (Fradkin, Nayak, Schoutens, 1999) (Goldman, Sohal, EF, 2019)

We will discuss bosons for simplicity, $\nu = \frac{k}{2}$

Consider k layers of bosons in a $\nu = \frac{1}{2}$ Laughlin state



$$\Psi_{1/2} \sim \prod_{i<j} (z_i - z_j)^2 \times \text{gaussians}$$

$\longleftarrow U(1)_2$

For each layer $\mathcal{L} = \frac{2}{4\pi} \varepsilon_{\mu\nu\lambda} a^\mu \partial^\nu a^\lambda + \dots$

$$\equiv \frac{2}{4\pi} a\,da + \frac{1}{2\pi} A\,da + \dots$$

Symmetry $\underbrace{U(1)_2 \times \dots \times U(1)_2}_{k \text{ factors}}$

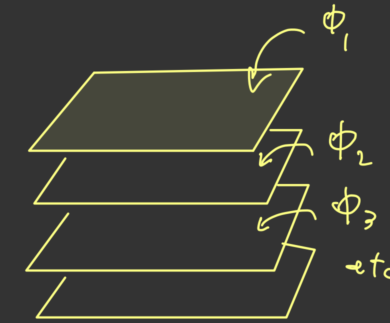
Chern-Simons level-rank duality $\quad U(1)_2 \longleftrightarrow SU(2)_1 \quad$ group is non-abelian

$I, \; e^{i\varphi/\sqrt{2}} \qquad j = 0, \frac{1}{2} \qquad$ the braids are abelian

Q: how to get to a state with non-abelian statistics?

Hint: somehow we need a theory on $SU(2)_k$

you need $U(1)_2 \times \cdots \times U(1)_2 \longrightarrow SU(2)_k$



$\langle \phi_j \, \phi_{j+1} \rangle \neq 0$

(A) (1) use the Chern-Simons level-rank duality

$SU(2)_1 \times \cdots \times SU(2)_1$

(2) construct a condensate $\rightarrow SU(2)_k$

The 1999 paper did this by condensing pairs of excitations on two layers at a time

$\Rightarrow$ Higgs (Meissner) mechanism projects onto a state with symmetry $SU(2)_k$ (clustering)

1999 was basically right (but not completely)

$\Rightarrow$ Dualities solve the problem

Web of Dualities

① Particle – Vortex duality (Peskin, 1978; Dasgupta & Halperin, 1981)

$$\to |D_A \Phi|^2 - m^2 |\Phi|^2 - u|\Phi|^4 \longleftrightarrow |D_b \varphi|^2 + m^2 |\varphi|^2 - u|\varphi|^4 + \frac{1}{2\pi} A db + \text{Maxwell}$$

external; dynamical

$$\boxed{J_\mu \longleftrightarrow \frac{1}{2\pi} \varepsilon_{\mu\nu\lambda} \partial^\nu b^\lambda}$$

; vortices $\longleftrightarrow$ particles

② Bosonization (Fradkin & Schaposnik, 1994; Seiberg, Senthil, Wang & Witten, 2016)

$$\bar\Psi (i \not{D}_A - M)\Psi - \frac{1}{8\pi} AdA \longleftrightarrow |D_a \phi|^2 - m^2 |\phi|^2 - u|\phi|^4 + \frac{1}{4\pi} ada + \frac{1}{2\pi} adA$$

Dirac fermion $\longleftrightarrow$ monopole ; $\boxed{\bar\Psi \gamma^\mu \Psi \longleftrightarrow \frac{1}{2\pi} \varepsilon^{\mu\nu\lambda} \partial_\nu a_\lambda}$

③ Fermion Particle – Vortex duality (Son, 2015; Metlitski & Vishwanath, 2016)

$$\bar\Psi (i\not{D}_A - M)\Psi - \frac{1}{8\pi} AdA \Leftrightarrow \bar\chi (i \not{D}_a + M)\chi + \frac{1}{8\pi} ada - \frac{1}{2\pi} adb + \frac{2}{4\pi} bdb - \frac{1}{2\pi} bdA$$

"QED_3"

Back to bosons at $\nu = \frac{1}{2}$:

$U(1)_2$

$$\mathcal{L}_A = |D_a \Phi|^2 - m^2 |\Phi|^2 - u|\Phi|^4 + \frac{2}{4\pi} a\,da + \frac{1}{2\pi} a\,dA$$

$$\langle \Phi \rangle = 0 \Rightarrow \quad \mathcal{L}_{eff} = -\frac{1}{2} \frac{1}{4\pi} A\,dA \Rightarrow \sigma_{xy} = \frac{1}{2}\left(\frac{e^2}{h}\right)$$

plateau transition

$$\langle \Phi \rangle \neq 0 \Rightarrow \langle a \rangle = 0 \Rightarrow \mathcal{L}_{eff} \sim \text{Maxwell} \Leftrightarrow \text{no Hall effect}$$

① Use the <u>fermionization</u> duality

$$\mathcal{L}_A \longleftrightarrow \mathcal{L}_B = \bar{\Psi}\, i \not{D}_b \Psi - \frac{3}{2}\frac{1}{4\pi} b\,db - \frac{1}{2\pi} b\,dA - \frac{1}{4\pi} A\,dA$$

$SU(2)_1$

② Aharony: v is an $SU(2)$ field

$$\mathcal{L}_B \leftrightarrow \mathcal{L}_C = |D_{v - \frac{1}{2}A\mathbb{1}}\, \phi|^2 - |\phi|^4 + \frac{1}{4\pi} \mathrm{Tr}\left(v\,dv - i\frac{2}{3}v^3\right) - \frac{1}{2}\frac{1}{4\pi} A\,dA$$

$\Rightarrow \mathcal{L}_A \longleftrightarrow \mathcal{L}_C$ and $U(1)_2 \longleftrightarrow SU(2)_1$

<u>but</u> Φ represents particles

ϕ represents monopoles

Non-Abelian States from Clustering

e.g Moore-Read state

$k=2$ layers of $\nu = 1/2$ (same for the other cases)

$$\mathcal{L}_A = \sum_{n=1,2} \left(|D_{a_n} \Phi_n|^2 - |\Phi_n|^4 \right) + \frac{2}{4\pi} a_n \, d a_n + \frac{1}{2\pi} A \, d(a_1 + a_2) \qquad (U(1)_2 \times U(1)_2)$$

$\updownarrow$

$$\mathcal{L}_B = \sum_{n=1,2} \left(|D_{u_n - A \mathbb{1} 1/2} \phi_n|^2 - |\phi_n|^4 \right) + \frac{1}{4\pi} \sum_n \mathrm{Tr}\left(u_n d u_n - \frac{2}{3} i u_n^3 \right) - \frac{1}{4\pi} A \, dA$$

"Pairing"

$$\mathcal{L}_\Gamma = \sum_{m,n} |\partial \Gamma_{mn} - i u_m \Gamma_{mn} + i \Gamma_{mn} u_n|^2 + V[\Gamma] - \sum_{m,n} \phi_m^\dagger \Gamma_{mn} \phi_n \quad \leftarrow \text{"pairing"}$$

$\langle \phi_1 \rangle = \langle \phi_2 \rangle = 0$ but $\langle \phi_1^\dagger \phi_2 \rangle \neq 0 \Rightarrow SU(2)_1 \times SU(2)_1 \to SU(2)_2$

ϕ_1 and ϕ_2 have charge $\underset{\sim}{Q = \frac{1}{2}}$ and $j = 1/2$ under $SU(2)_2 \Rightarrow$ non-abelian statistics

$\chi_n^a = \underset{\sim}{\phi_n} t^a \phi_n \quad \Leftrightarrow$ Majorana fermions

Construction of a Fibonacci FQH state (Goldman, Sohal, EF, 2020)

* Want a FQH state with only Fibonacci anyons

$$\tau * \tau = \mathbb{1} + \tau \quad \text{(and no other anyons)}$$

$\Rightarrow$ universal quantum computing (3 τ's form a qubit)

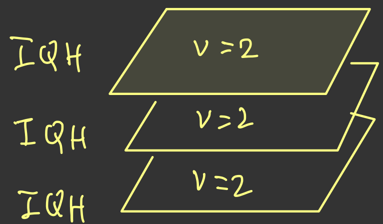
* Topological QFT?

$$(G_2)_1 \longleftrightarrow U(2)_{3,1} = \frac{SU(2)_3 \times U(1)_2}{\mathbb{Z}_2}$$

$$\mathcal{L}_{Fib} = \frac{3}{4\pi} \mathrm{Tr}\left[\underset{\substack{\uparrow \\ SU(2) \text{ gauge field}}}{a}\, d a - \frac{2}{3} i a^3\right] - \frac{1}{4\pi} \mathrm{Tr}[a]\, d\, \underset{\substack{\uparrow \\ U(1)_2}}{\mathrm{Tr}[a]} + \frac{1}{4\pi} \underset{\substack{\uparrow \\ \text{background}}}{A}\, d\, \mathrm{Tr}[a]$$

$$\Rightarrow \quad \nu = 2 \qquad \left(\sigma_{xy} = 2 \frac{e^2}{h} \right)$$

* Start with 3 layers of Diracs at $\nu = 2 \to 1$ transition (IQH)



$$\mathcal{L} = \sum_{n=1}^{3} \left[\bar{\Psi}_n (i \not{D}_A - M) \Psi_n - \frac{3}{2} \frac{1}{4\pi} A\, dA \right]$$

Parity anomaly

$$D_A = \partial - iA$$

Duality: Free Dirac $\Psi \Longleftrightarrow$ Wilson–Fisher boson ϕ + $U(N)_1$

OK since $U(N)_1 \Longleftrightarrow \mathcal{L}_{eff} = -\frac{N}{4\pi} A\, dA$ (trivial)

* Set $N = 2$

$$\mathcal{L} = \sum_n \left[|D_{a_n} \phi_n|^2 - r|\phi_n|^2 - |\phi|^4 + \mathcal{L}_{CS}[a_n] \right] + \frac{1}{2\pi} A\, d\, Tr[a_1 - a_2 + a_3]$$

* Clustering: $\langle \Gamma_{mn} \rangle = \langle \phi_m^\dagger \phi_n \rangle \neq 0 \quad (m \neq n)$, $\langle \phi_n \rangle = 0$

$\Rightarrow$ pins $a_1 = a_2 = a_3 \equiv a \Rightarrow \frac{1}{2\pi} A\, d\, Tr[a_1 - a_2 + a_3] \equiv \frac{1}{2\pi} A\, d\, Tr[a]$

* The physical densities are pinned $\rho_1 = -\rho_2 = \rho_3$

$\Rightarrow$ layer exchange symmetry is broken

$$\Rightarrow \mathcal{L}_{U(2)_3} = 3\,\mathcal{L}_{CS}[a] + \frac{1}{2\pi} A\, d\, \mathrm{Tr}[a]$$

* To get Fibonacci $\Leftrightarrow$ attach a unit of flux to the fermions

$\Rightarrow$ fermions $\rightarrow$ bosons

flux attachment: $3\,\mathcal{L}_{CS}[a] + \frac{1}{2\pi} b\, d\, \mathrm{Tr}[a] + \frac{1}{4\pi}(b+A)\, d\,(b+A)$

fluctuating U(1) gauge field

* Integrating out $b_\mu \Rightarrow$ obtain $\mathcal{L}_{Fib}$!

$\Rightarrow$ interpret $\underline{\phi^\dagger t^a \phi}$ as the Fibonacci anyon τ