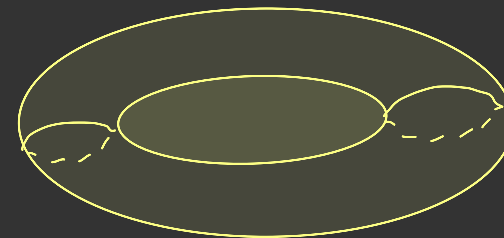
L2 10/2/23

Vacuum Degeneracy

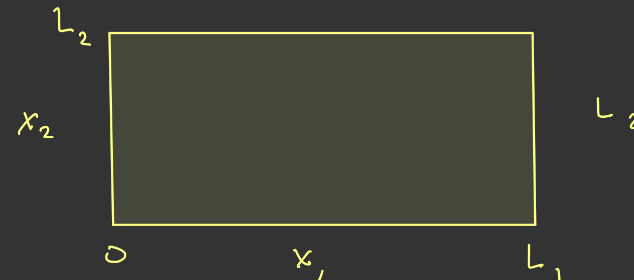
(1)

$f_{\mu\nu} = 0$



torus T^2

$$\left.\begin{aligned} \int_0^{L_1} dx_1\, a_1 &\equiv \bar{a}_1 \\ \int_0^{L_2} dx_2\, a_2 &\equiv \bar{a}_2 \end{aligned}\right\} \text{gauge invariant}$$

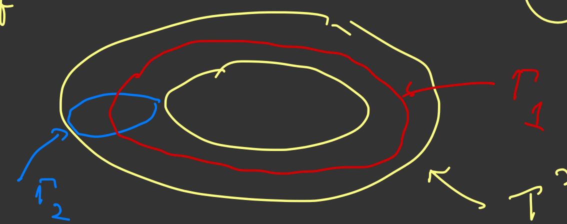


$$a_1 = \partial_1 \varphi + \frac{\bar{a}_1}{L_1}, \quad a_2 = \partial_2 \varphi + \frac{\bar{a}_2}{L_2}$$

$$S = \frac{k}{4\pi} \int dx_0\, \varepsilon_{ij}\, \bar{a}_i\, \partial_0 \bar{a}_j$$

$$\Rightarrow [\bar{a}_1, \bar{a}_2] = i\frac{2\pi}{k} \Longleftrightarrow \bar{a}_2 = -i\frac{2\pi}{k}\frac{\partial}{\partial \bar{a}_1}$$

Wilson loops on non-contractible cycles of T^2



$$W(\Gamma_1) = e^{i\int_0^{L_1} a_1} \equiv e^{i\bar{a}_1}$$

$$W(\Gamma_2) = e^{i\int_0^{L_2} a_2} \equiv e^{i\bar{a}_2}$$

$\Rightarrow$ $$W(\Gamma_1)\, W(\Gamma_2) = e^{-i\frac{2\pi}{k}}\, W(\Gamma_2)\, W(\Gamma_1)$$

$\Rightarrow$ invariance under large g.t. $\Rightarrow$ $\bar{a}_1, \bar{a}_2$ define a 2-torus

$$U_2 = e^{ik\bar{a}_2} \quad , \quad U_2 = e^{-ik\bar{a}_1} \quad \Rightarrow \quad U_1 U_2 = e^{i\frac{2\pi}{k}}\, U_2 U_1$$

$\Rightarrow$ U_1 and U_2 shift $\bar{a}_1$ and $\bar{a}_2$ by 2π

$$U_1^{-1}\, W(\Gamma_1)\, U_1 = W(\Gamma_1) \quad ; \quad U_2^{-1}\, W(\Gamma_2)\, U_2 = W(\Gamma_2)$$

Let $|0\rangle$ / $W(\Gamma_1)|0\rangle = |0\rangle$ (e.v. $= 1$)

$\Rightarrow$ $W(\Gamma_2)|0\rangle$ has e.v. $e^{-i\frac{2\pi}{k}}$

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$$W(\Gamma_1)\, W^p(\Gamma_2)\,|0\rangle = e^{-i 2\pi p/k}\, W^p(\Gamma_2)\,|0\rangle$$

$\Rightarrow$ We have k linearly indep. states on a torus

on a surface Σ of genus g the degeneracy is g^k

$\Rightarrow$ We have a finite-dim. Hilbert space

This theory is known as the C. S. theory $U(1)_k$ ← level

$\Rightarrow$ Laughlin states have a degeneracy m on a torus

One can construct wave functions for these states by replacing the coordinates z_i by Theta-functions $\Theta(z_i)$

Multi component Chern - Simons theory

We will have a_μ^I with $I = 1, \dots, N$, gauge group : $U(1)^N$

$$\mathcal{L}(a_\mu^I, A_\mu) = -\frac{1}{4\pi} K_{IJ}\, \varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda - \frac{e}{2\pi} A_\mu t_I \varepsilon^{\mu\nu\lambda} \partial_\nu a_\lambda^I + j^\mu_{qp}\, l_I\, a_\mu^I$$

$K_{IJ} = K_{JI}$; $K_{11} \equiv m$ (odd for fermions)

t_I : charge vector, l_I : qp vector

worldlines of excitations (qp's)

<u>vacuum deneracy</u> : $|\det K|^g$ (g = genus)

<u>Filling fraction</u> : $\nu = (K^{-1})_{IJ} t_I t_J$

<u>Charge</u> $Q = -e\, (K^{-1})_{IJ} t_I l_J$; <u>Statistics</u> : $\theta = \pi (K^{-1}) l_I l_J$

<u>mutual statistics</u> $\theta(l, l') = \pi (K^{-1})_{IJ} l'_I l_J$

Short Detour: BF Theory and Toric Codes

* Toric Code is another name for the deconfined phase of a $\mathbb{Z}_2$ (and $\mathbb{Z}_N$) lattice gauge theory in 2+1 dimensions
* Its topological content is captured by the BF theory
* It has two gauge fields, a_μ and b_μ

$$\mathcal{L} = \frac{N}{2\pi} \varepsilon_{\mu\nu\lambda} b^\mu \partial^\nu a^\lambda \quad ; \quad K\text{-matrix} \quad K = \begin{pmatrix} 0 & N \\ N & 0 \end{pmatrix}, \quad N \in \mathbb{Z}$$

degeneracy: $|\det K| = N^2$

$N = 2$: $\mathbb{Z}_2$ toric code; It has 4 anyons: I, e, m, $\psi = em$

$$W_\gamma^e = e^{i \oint_\gamma dx_\mu a^\mu}, \quad W_{\tilde{\gamma}}^m = e^{i \oint_{\tilde{\gamma}} dx_\lambda b^\mu} \quad (\gamma, \tilde{\gamma} \text{ are linked})$$

$\{W_\gamma^e, W_{\tilde{\gamma}}^m\} = 0$: e and m are bosons but anticommute ; ψ is a Majorana fermion

Generalizations : Multi-Component states

* Almost all FQH states are abelian (i.e. multi-component Laughlin and Jain)

$$\Psi_{mmn}[\{z_i\},\{w_j\}] \sim \prod_{i<j} (z_i - z_j)^m (w_i - w_j)^m \prod_{i \le j} (z_i - w_j)^n \times \text{exponentials}$$

→ Halperin (m,m,n) states ; $\nu = \frac{2}{m+n}$

* $n = m-1 \Rightarrow$ spin singlets
* m odd for fermions, m even for bosons
* All abelian fermionic states have odd denominators
* Effective field theory : K-matrix (2×2)

$$K = \begin{pmatrix} m & n \\ n & m \end{pmatrix}_{IJ}, \quad \text{charge vector } t_I = (1,1), \quad l_I = (1,0),(0,1)$$

qp charges $Q = \pm \frac{e}{m+n}$, statistics $\theta = \pi \frac{m}{m^2-n^2}$

degeneracy on a torus $= |\det K| = |m^2 - n^2|$

Prominent examples

(331), $\nu = 1/2$, $K = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$, degeneracy $= 8$, $Q = \pm \frac{e}{4}$, $\theta = \frac{3\pi}{8}$, P

(332), $\nu = 2/5$, $K = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$, degeneracy $= 5$, $Q = \pm \frac{e}{5}$, $\theta = \frac{3\pi}{5}$, S

(112), $\nu = 2/3$, $K = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, degeneracy $= 3$, $Q = \pm \frac{e}{3}$, $\theta = -\pi/3$, S

charge and neutral modes: $a^\mu_\pm = \frac{1}{\sqrt{2}} (a^\mu_1 \pm a^\mu_2)$, $l_\pm = l_1 \pm l_2$

$$\mathcal{L} = \frac{m+n}{4\pi} a_+ d a_+ - \sqrt{2} \frac{e}{2\pi} A d a_+ + \frac{m-n}{4\pi} a_- d a_- + j^{qp}_\mu \left[\frac{1}{\sqrt{2}} l_+ a^\mu_+ + \frac{1}{\sqrt{2}} l_- a^\mu_- \right]$$

$\Rightarrow$ a_+ is the charged mode, a_- is the neutral mode

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Edge States: charged edge mode + neutral edge mode

$$\mathcal{L} = \frac{m+n}{4\pi} (\partial_0 \varphi_+ \partial_1 \varphi_+ - v_c (\partial_1 \varphi_+)^2] + \frac{|m-n|}{4\pi} (\partial_0 \varphi_- \partial_1 \varphi_- - s\, v_n (\partial_1 \varphi_-)^2)$$

$s = \text{sign}(m-n)$

φ_+ : charged, φ_- : neutral, $v_c > v_n$

CFT representation of FQH wave functions

* All abelian states can be written as expectation values in a 2D CFT

e.g. Laughlin $\Psi_m \sim \left\langle \prod_{i=1}^{N} e^{i\sqrt{m}\,\varphi(z_i)} \, e^{-\int d^2z' \sqrt{m}\, \rho_0 \varphi(z')} \right\rangle$ areal charge density

where $\varphi(z)$ is a compactified chiral boson in Euclidean space (imag. time)

$\varphi \sim \varphi + 2\pi\sqrt{m}$ (here I rescaled φ by $\sqrt{m}$ and $R = \sqrt{m}$)

* electron operator $V_e \sim e^{i\sqrt{m}\,\varphi(z)}$ scaling dimension $\Delta_e = \frac{m}{2}$
* quasi-hole (vortex) $V_{qh} \sim e^{i\varphi(z)/\sqrt{m}}$ " " $\Delta_{qp} = \frac{1}{2m}$
* physical states must be local w.r.t. the electron

$\Rightarrow$ only V_{qh}^{n} $(n = 1, \dots, m)$ are allowed

$\Rightarrow$ The ideal bulk wave functions are correlators of an Euclidean CFT!
This is the same CFT of the edge but in imaginary time.