Geometry of tropical varieties

Lecture 3

30/06/2022

Summary

Fan in $\mathbb{R}^n$ $\quad \Sigma = \{\sigma\}$ Collection of Cones in $\mathbb{R}^n$

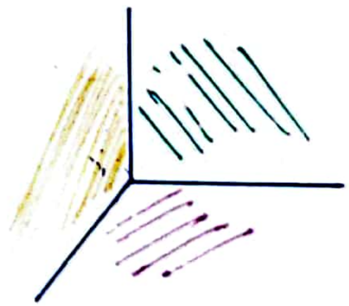
- $\sigma \in \Sigma$, τ face of $\sigma \Rightarrow \tau \in \Sigma$.
- $\sigma, \eta \in \Sigma$, $\sigma \cap \eta$ common face of σ and η.

Fanfold $\quad Y \subseteq \mathbb{R}^n$. support of a (rational) fan, $\quad Y = \bigcup_{\sigma \in \Sigma} \sigma$

Tropical fan $\quad \Sigma = \{\sigma\}$ Pure dimensional, balancing condition

$$\forall \tau \in \Sigma_{d-1}, \quad \sum_{\sigma \supsetneq \tau} e_{\sigma/\tau} = 0 \quad \text{in } \mathbb{R}^n / N_\tau$$

Tropical fanfold $\quad$ support of a tropical fan



Example

- Bergman fans are tropical.
- Complete fans are tropical.

§6. Geometry of tropical fanfolds

Ref A.-Piquerez, Homology of tropical fans.

Fans are all unimodular.

§ Operations and constructions on tropical fans and fanfolds

(1) Product

Σ_1 tropical fan in $\mathbb{R}^{n_1}$

Σ_2 tropical fan in $\mathbb{R}^{n_2}$

$\Sigma_1 \times \Sigma_2 := \{ \sigma_1 \times \sigma_2 \mid \sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2 \}$ is tropical.

$\Rightarrow$ Products of tropical fanfolds is tropical.

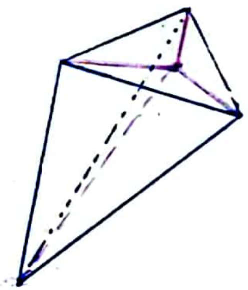
(2) Star subdivision

Σ tropical fan

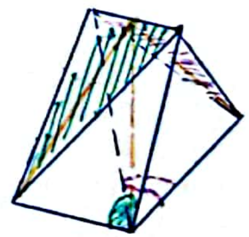
$\sigma \in \Sigma$

ρ rational ray in the interior of σ

$\Sigma_{(\rho)}$ star subdivision remains tropical.



Example 1



Example 2

(3) Divisors on tropical fans

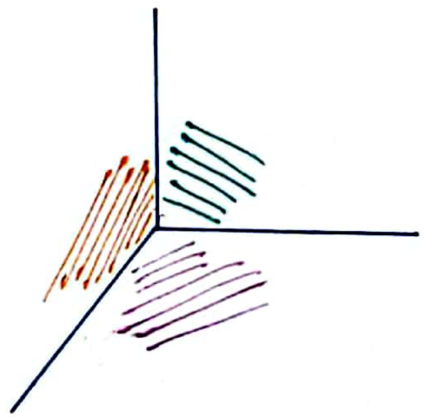
Σ a tropical fan of dimension d.

A <u>divisor</u> on Σ is a (weighted) subfan of Σ of dimension $d-1$ which verifies the balancing condition.

$\omega : \Sigma_{d-1} \to \mathbb{Z}$ such that $\forall\, \tau$ of dim $d-2$

$$\sum_{\substack{\eta \ni \tau \\ \text{of dim } d-1}} \omega(\eta)\, e_{\eta/\tau} = 0 \quad \text{in } \mathbb{R}^n / N_\tau .$$

Δ_{d-1} = Support of ω



Construction

Σ tropical fan $\qquad Y$ corresponding fanfold

$f : Y \to \mathbb{R}$ continuous, conewise integral linear.

$\sigma \in \Sigma \qquad f_\sigma : \sigma \to \mathbb{R}$

$\eta \in \Sigma_{d-1}$, $\omega(\eta)$

$$\omega(\eta) := -\sum_{\sigma \ni \eta} f_\sigma(e_{\sigma/\eta}) + f_\eta\Big(\sum_{\sigma \ni \eta} e_{\sigma/\eta}\Big).$$

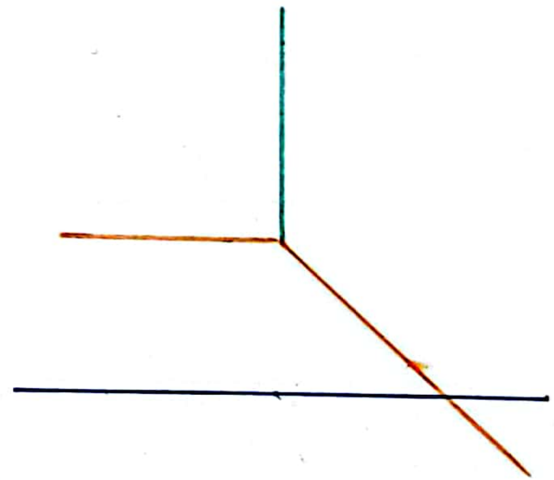
<u>Fait</u> $\omega: \Sigma_{d-1} \to \mathbb{Z}$ is a divisor. That is, it verifies the balancing condition.

<u>Definition</u> (Principal divisor)

$f: Y \to \mathbb{R}$ conewise integral linear

$\mathrm{div}(f)$ the corresponding divisor.

<u>Definition</u> . A divisor is called reduced if ω takes values 0 or 1.



(4) <u>Tropical modification</u>

Y tropical fanfold

Σ fan structure

$f: Y \to \mathbb{R}$ conewise linear

$\mathrm{div}(f)$ reduced

$\Rightarrow \tilde{Y}, \tilde{\Sigma}$
tropical modification

- Take the graph of f : $\Gamma_f \subseteq \mathbb{R}^n \times \mathbb{R}$
- For any η in the support of $\mathrm{div}(f)$ add a new cone $\eta \times \mathbb{R}_+$

Proposition

Tropical modification is a tropical fan.

Notation

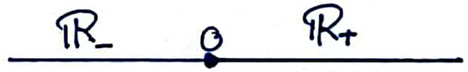
$TH_f(\Sigma)$

$TH_f(Y)$ corresponding fanfold

$\mathrm{div}(f)$ is called center of the modification.

(5) Shellability for tropical fans and tropical fanfolds

Definition A tropical fan is called shellable if it can be obtained from the above operations starting from the fan



- We allow to take inverse of the star subdivision.
- In tropical modification, we require the center of the modification to be itself shellable

A tropical fanfold is called shellable if it is the support of a shellable tropical fan.

Examples

- $\mathbb{R}^n$ is a shellable tropical fanfold.
- For any matroid M, the Bergman fanfold Y_M, support of M, is shellable. (Proved first in Shaw's thesis: based on deletion - contraction in matroids)
- There exist shellable tropical fanfolds which are not Bergman.

Theorem (A.-Piquerez)

- Many properties of tropical fans are shellable.
- shellable tropical fanfolds are smooth.
- Shellable tropical fans enjoy the geometric properties of complex projective manifolds.
- Shellable tropical fans provide local charts for tropical varieties reminiscent of complex projective manifolds.

§ Chow rings of tropical fans

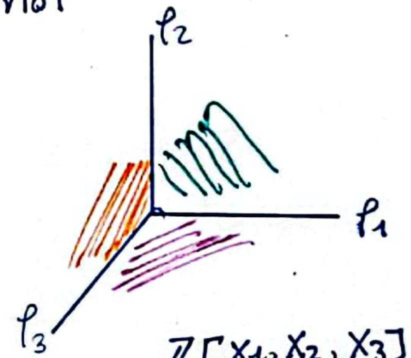
Σ tropical fan in $\mathbb{R}^n$.
d dimension of Σ

Definition (Chow ring $A^\cdot(\Sigma)$)

Consider the polynomial ring $\mathbb{Z}[x_\rho]_{\rho\in\Sigma_1}$, given by rays of Σ.

Let

- I ideal generated by monomials of the form $\prod_{\rho\in A} x_\rho$ for $A \subseteq \Sigma_1$ such that the rays $\rho\in A$ do not form a cone in Σ.



- J ideal generated by linear forms:
  $\forall\ \underset{\text{linear}}{l : \mathbb{Z}^n \to \mathbb{Z}}$, $\sum_{\rho\in\Sigma_1} l(e_\rho)\, x_\rho \in J$

$$\mathbb{Z}[x_1, x_2, x_3]\Big/\begin{pmatrix} x_1 - x_2, \\ x_1 - x_3, \\ x_1x_2x_3 \end{pmatrix} = \mathbb{Z} \oplus \mathbb{Z}x_1 \oplus \mathbb{Z}x_1^2$$

The chow ring of Σ: $A^\cdot(\Sigma) := \mathbb{Z}[x_\rho]_{\rho\in\Sigma_1}\big/ I + J$.

$$A^{\bullet}(\Sigma) = A^{0}(\Sigma) \oplus A^{1}(\Sigma) \oplus A^{2}(\Sigma) \oplus \cdots$$

is graded.

Basic properties

- For $k \geq d+1$, $A^{k}(\Sigma) = (0)$.
- $A^{0}(\Sigma) = \mathbb{Z}$.
- $A^{k}(\Sigma)$ is generated by square-free monomials.
- There is a degree map
$$\deg : A^{d}(\Sigma) \to \mathbb{Z}$$
$$\sum a_{\sigma} \prod_{\rho \in \sigma} x_{\rho} \mapsto \sum a_{\sigma}.$$
- Induces a pairing
$$A^{k}(\Sigma) \times A^{d-k}(\Sigma) \to A^{d}(\Sigma) \to \mathbb{Z}$$
$$(a, b) \longmapsto a.b \mapsto \deg(ab)$$

Remark $A^{\bullet}(\Sigma)$ is the chow ring of the toric variety $\mathbb{P}_{\Sigma}$. $\mathbb{P}_{\Sigma}$ is not compact if $Y \neq \mathbb{R}^{n}$.

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Theorem (A.-Piquerez)

Assume Σ is shellable. (Unimodularity is always assumed)

Then, we have

- (Poincaré duality) the pairing $A^k(\Sigma) \times A^{d-k}(\Sigma) \to \mathbb{Z}$ is perfect.

- (Hodge theory) Assume in addition that Σ is convex. That means there exists $l: Y \to \mathbb{R}$ conewise integral linear and convex. Let $\omega = \sum_{\rho \in \Sigma_1} l(e_\rho)\, x_\rho \in A^1(\Sigma)$. Then,

  (Hard Lefschetz) For $k \leq d/2$, $\omega^{d-2k} : A^k(\Sigma) \longrightarrow A^{d-k}(\Sigma)$, $a \longmapsto \omega^{d-2k} \cdot a$ is an isomorphism.

  (Hodge-Riemann) Let $P^k := \operatorname{Ker}(\omega^{d-2k+1} : A^k(\Sigma) \longrightarrow A^{d-k+1}(\Sigma))$. The pairing $P^k \times P^k \to \mathbb{Z}$, $(a, b) \longmapsto (-1)^k \deg(a \cdot b \cdot \omega^{d-2k})$ is positive definite.

<u>Proof</u> The listed properties are shown to be shellable.

The theorem follows from the basic case $\underline{\mathbb{R}_- \quad 0 \quad \mathbb{R}_+}$

$\square$

<u>Corollary</u> (Hodge theory for matroids)

Adiprasito-Huh-Katz 2015

Chow ring of matroids verify Poincaré duality, Hard Lefschetz, Hodge-Riemann.

<u>Proof</u>

- $A^{\cdot}(M) := A^{\cdot}(\Sigma_M)$
- Σ_M shellable $\Rightarrow$ Poincaré duality holds.
- Σ_M is convex $\Rightarrow$ Hodge theory package holds.

§ Application: Log-concavity

M matroid on the ground set $E = \{0, 1, \dots, m\}$.

Characteristic Polynomial $f_M(x) = \sum_{A \subseteq E} (-1)^{|A|} x^{r - r(A)}$.

$$= a_0 x^r - a_1 x^{r-1} + \cdots + (-1)^r a_r.$$

$a_0, \dots, a_r \geq 0$.

Example $G = (V, E)$ graph connected

M matroid on E Bases = Spanning trees.

f_M = chromatic polynomial value at k = # proper vertex colorings using $\leq k$ colors.

Theorem (Adiprasito-Huh-Katz 2015)

The sequence $a_0, a_1, \dots, a_r$ is log-concave. $a_i a_{i+2} \leq a_{i+1}^2$.

Proof. $h(x) = \dfrac{f(x)}{1-x} = \sum_{j=0}^{r-1} b_j x^{r-1-j}$

- It will be enough to show log-concavity of b_j.
- $b_j = \deg(\alpha^j \beta^{r-1-j})$ $\qquad \alpha = \sum_{F \ni e} x_F \qquad \beta = \sum_{F \not\ni e} x_F$
- $\det \begin{pmatrix} (\alpha,\alpha) & (\alpha,\beta) \\ (\alpha,\beta) & (\beta,\beta) \end{pmatrix} \leq 0$ using Hodge-Riemann.
  $\Rightarrow$ Log-concavity for the initial terms.
- General case: Truncation of M. $\square$