SMALL DEPTH CIRCUIT LOWER BOUNDS VIA THE SWITCHING LEMMA

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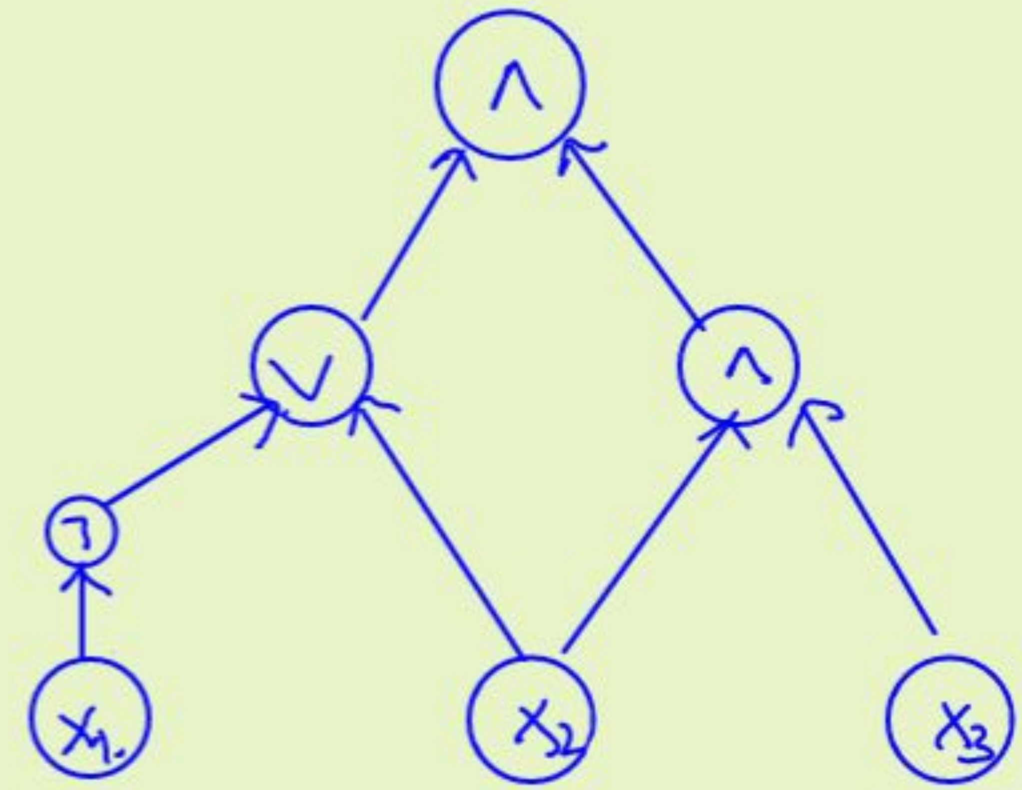
Circuit Complexity

→ Boolean circuit

De Morgan basis

AND / OR / NOT



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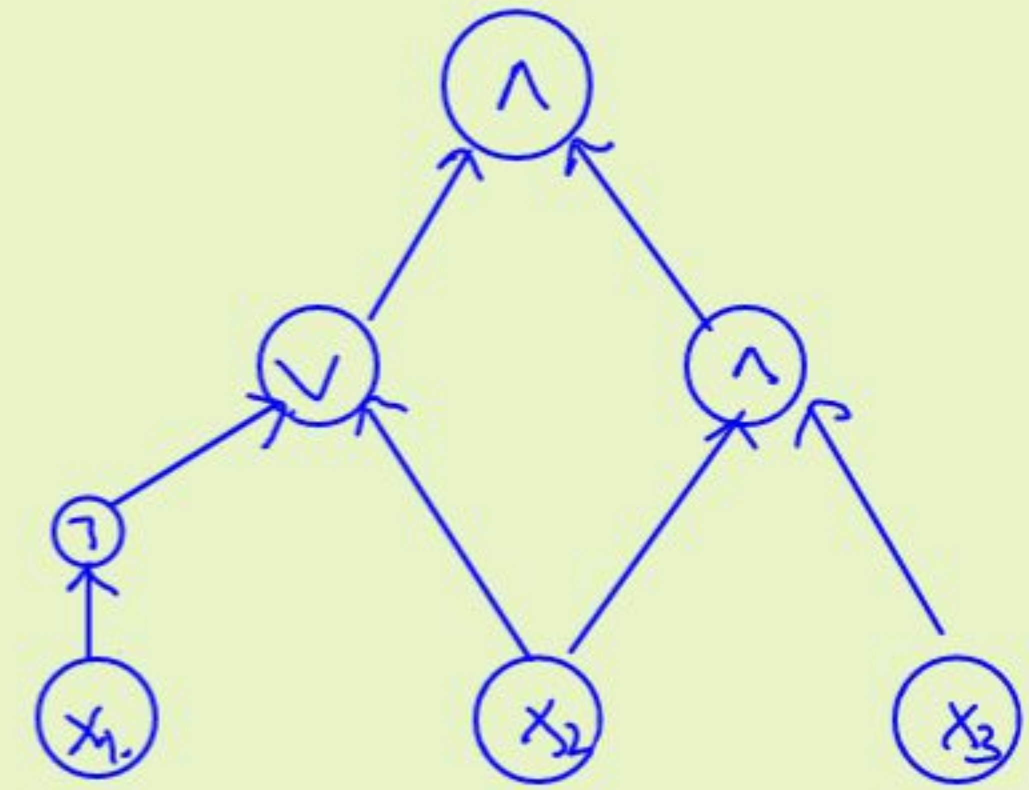

Size = # of gates = 6

Depth = longest AND/OR path = 2.

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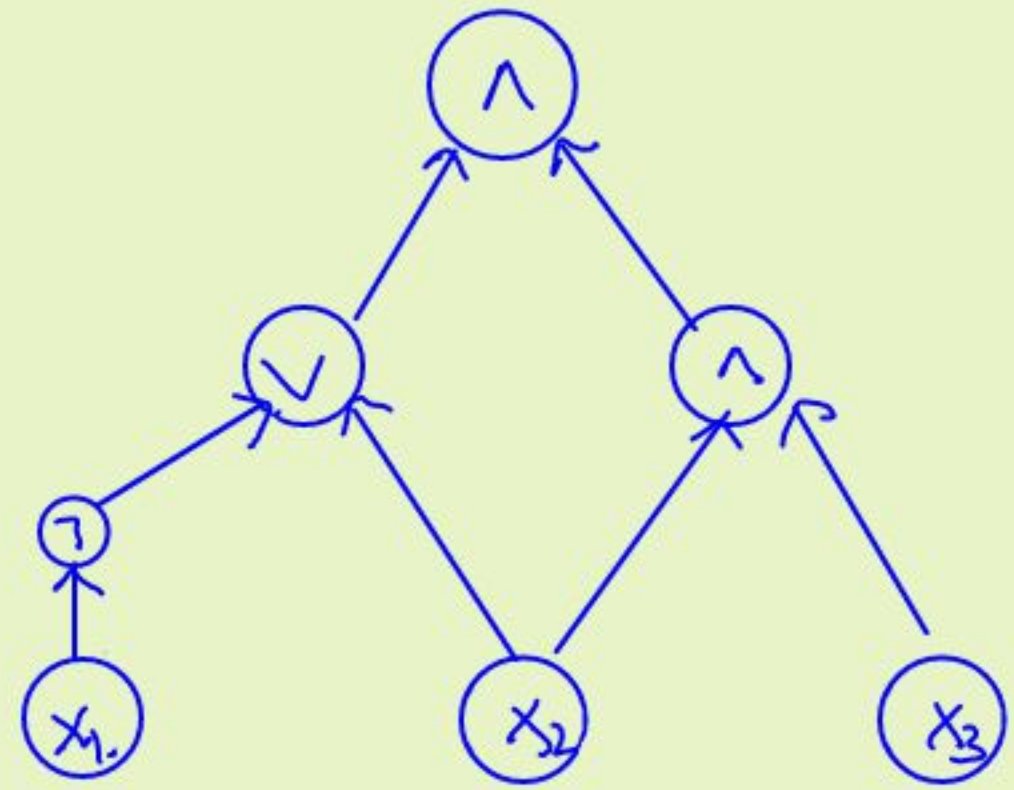

→ Computes $f : \{0,1\}^n \to \{0,1\}$

→ Any f has a circuit of size $O(2^n)$.

→ Most f need size $2^{\Omega(n)}$

→ Explicit f ?

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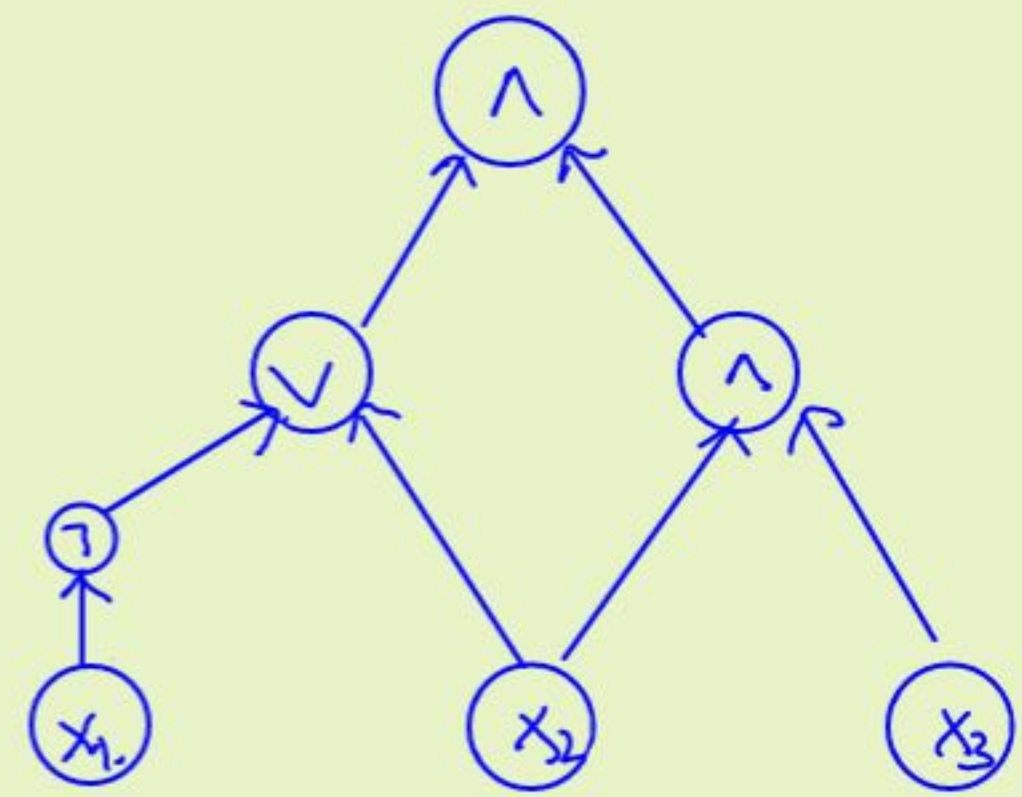

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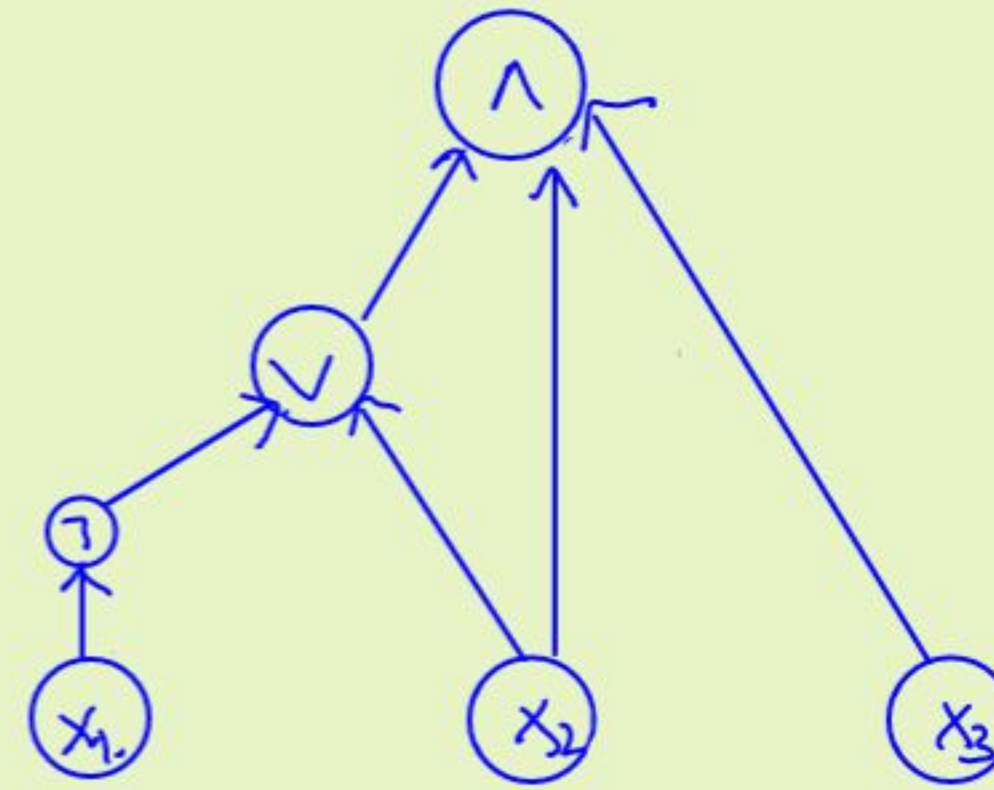

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$$Parity_n = \oplus_n : \{0,1\}^n \to \{0,1\}$$

$$(x_1, \dots, x_n) \mapsto x_1 \oplus \dots \oplus x_n$$

Cannot fix to a constant without setting all inputs.

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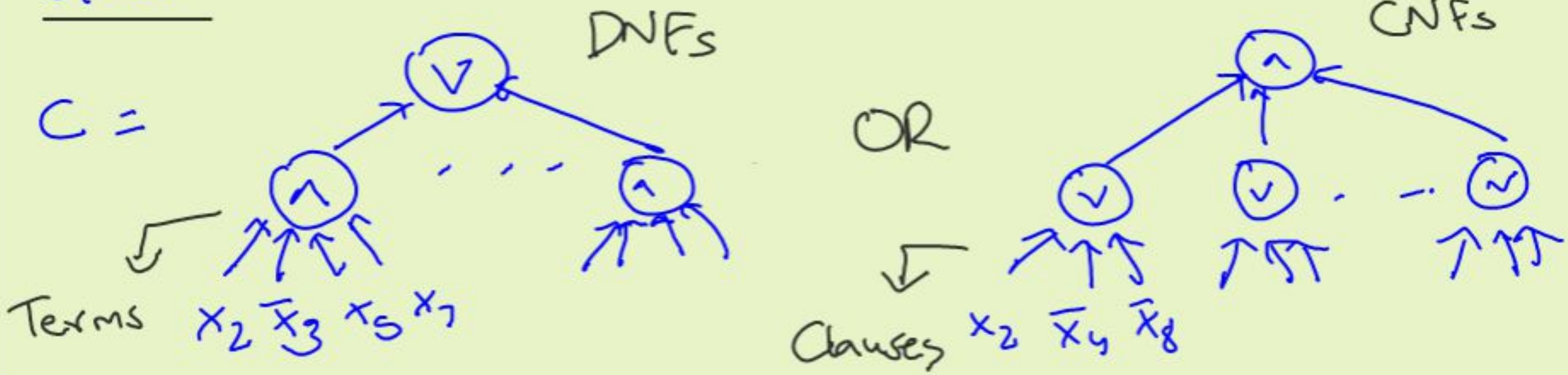
Thm: [FSS'81, Ajtai'83, ..., Håstad'86] Depth d circuits for $\oplus_n$ need size $\exp(\Omega(n^{1/d-1}))$.

2	n
3	$\sqrt{n}$
4	$n^{1/3}$

Ex: Tight!

$\underline{d=2}$

$C =$



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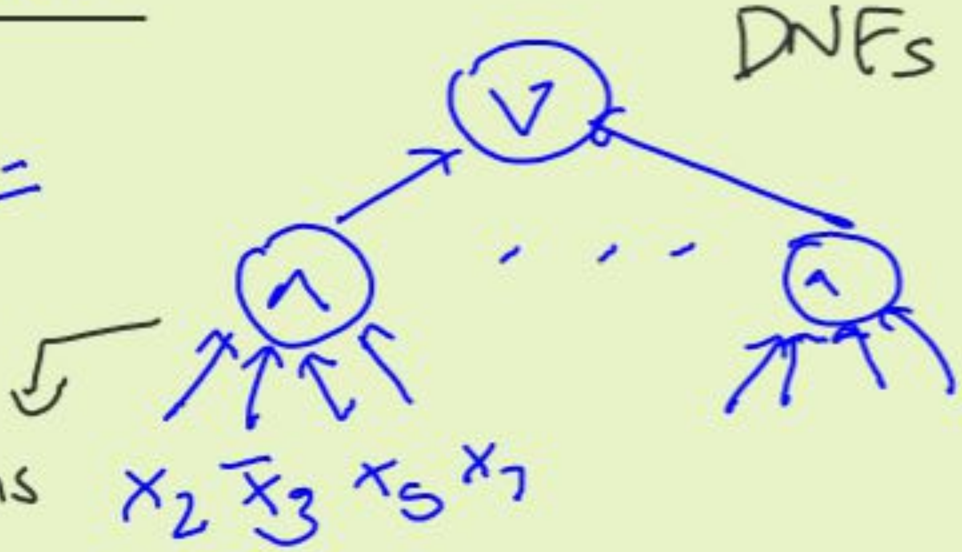
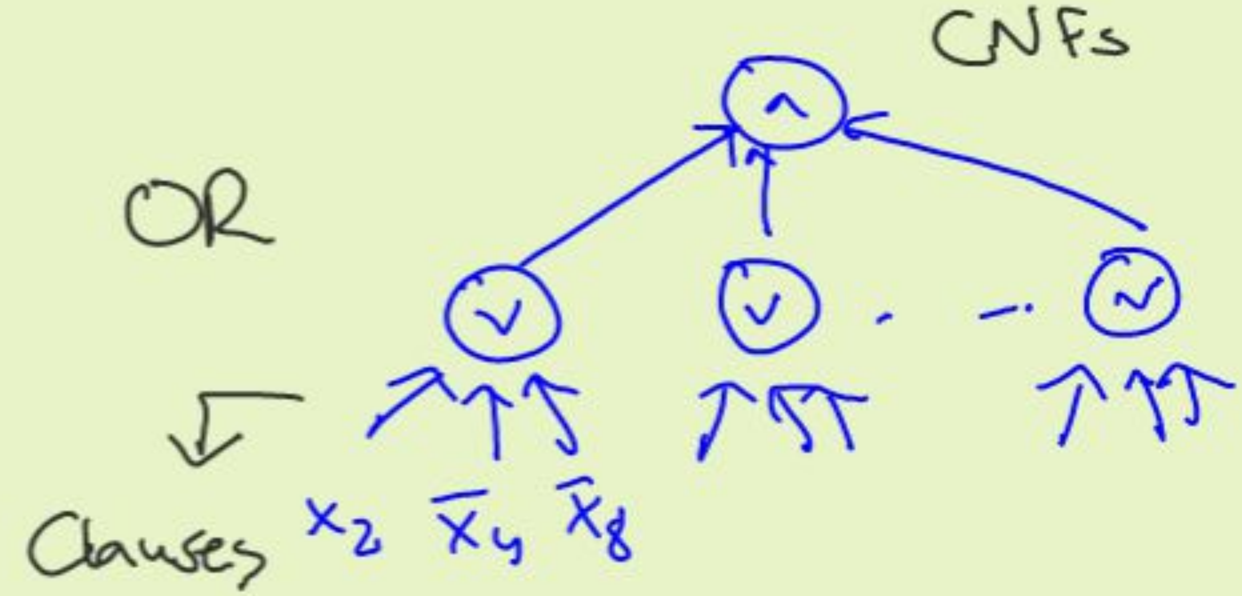


OR



Clm: C a DNF for $\oplus_n \implies$ all terms have width n

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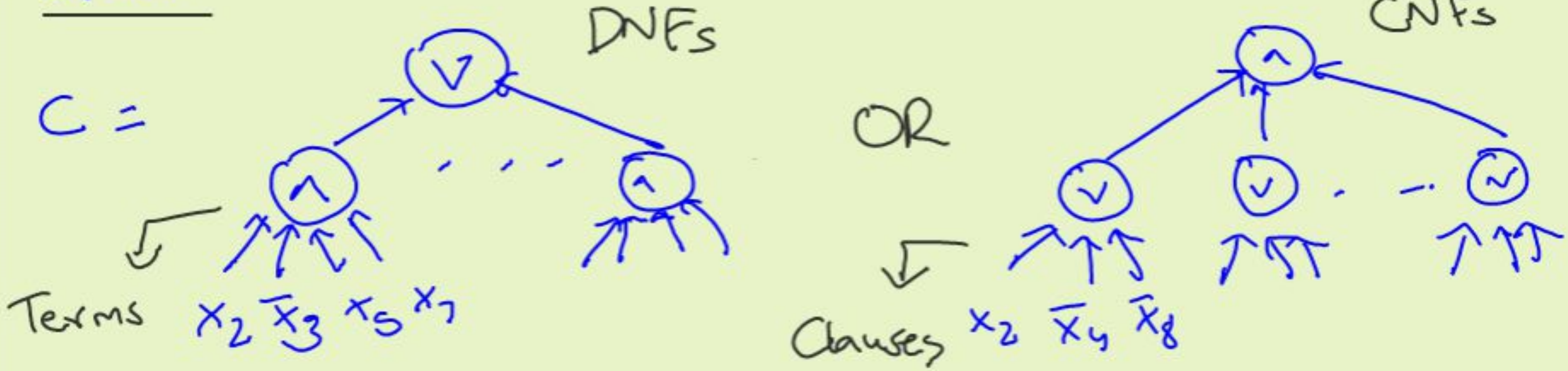

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Obs: $\oplus_n^{-1}(1) = 2^{n-1} \Rightarrow 2^{n-1}$ terms needed.

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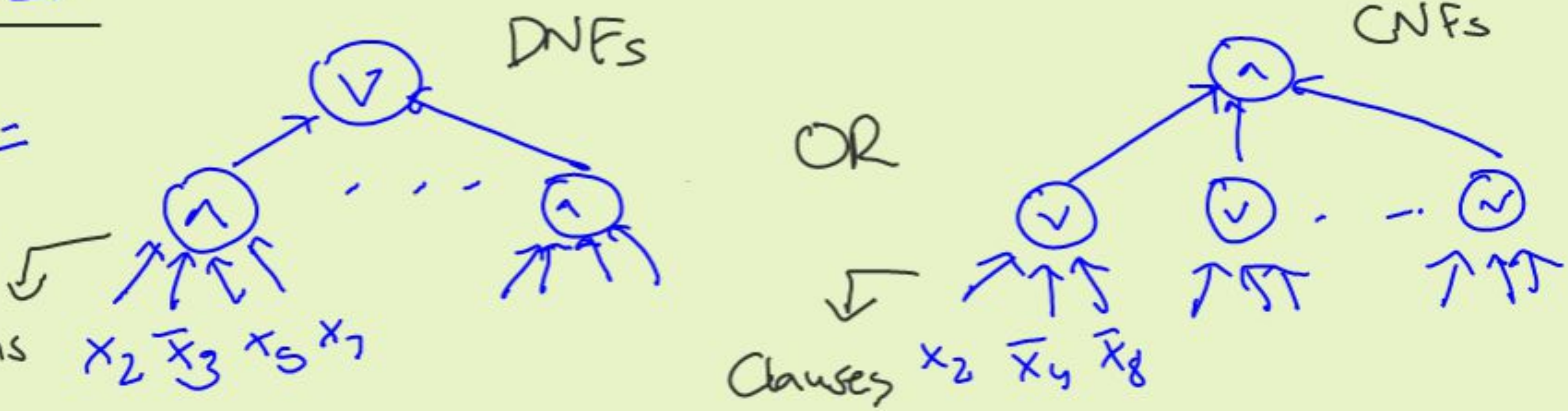

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[Analogous argument for CNFs]

Higher depths

Idea: Reduce to depth-2 iteratively via random restrictions.

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Random restriction with $*$-probability $q \in [0,1]$

→ Choose $S \subseteq [n]$, $|S| = qn$ u.a.r.

→ Set $\rho(x_i) = \begin{cases} * & i \in S \\ \text{uniformly random bit} & i \notin S \end{cases}$

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$\rho \sim R_{qn}$.

Informal Switching lemma

"Lemma": Any small-width DNF/CNF F simplifies whp under random restrictions.

$$\Pr_{\rho \sim R_{an}} [\underbrace{F\restriction_\rho}_{\text{restricted function}} \text{ "complicated"}] \leq \text{small}.$$

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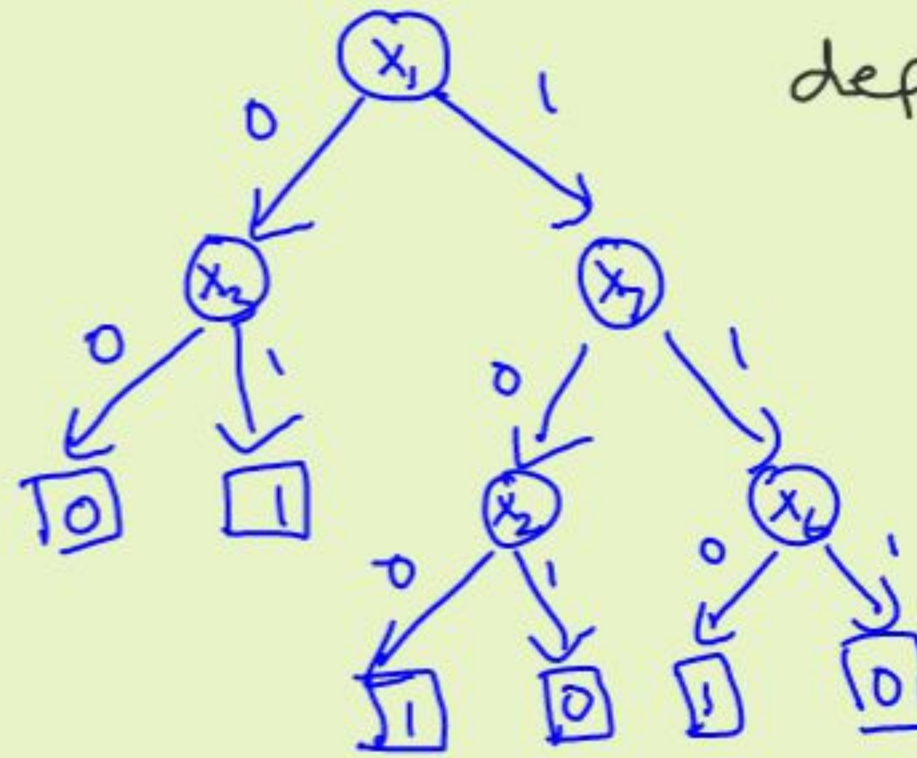
→ Want: q not too small (E.g. $q > 1/n$)

General Switching Lemma

Decision trees

DTs of depth k



depth $= 3$

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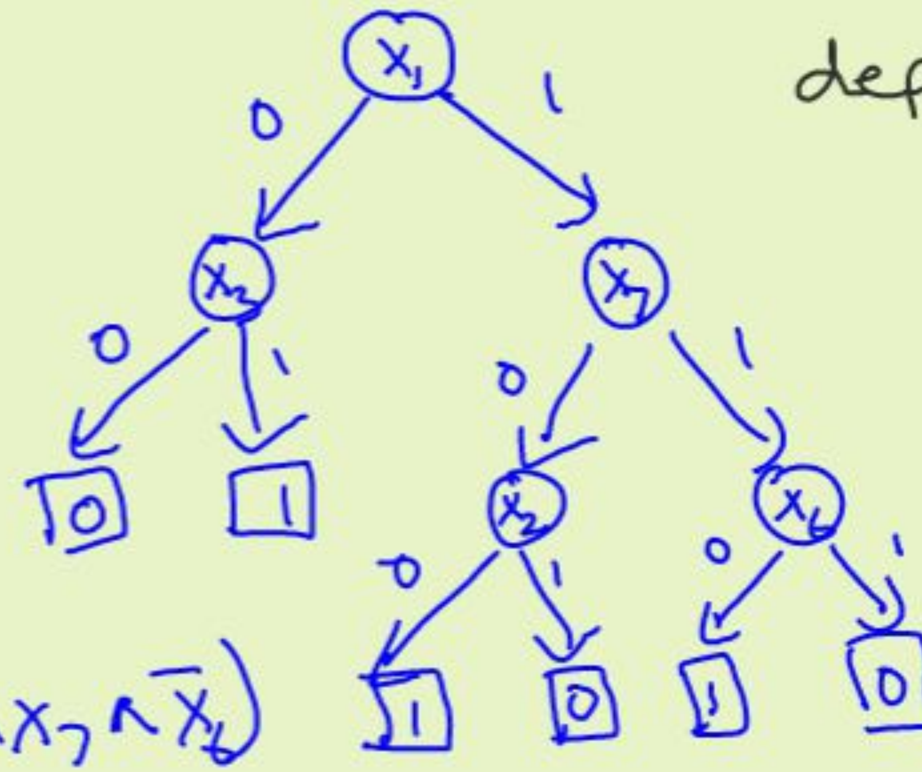
k-DNFs $\qquad$ k-CNFs



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Eg: $(\bar{x}_1 \wedge x_3) \vee (x_1 \wedge \bar{x}_7 \wedge x_2) \vee (x_1 \wedge x_7 \wedge \bar{x}_6)$

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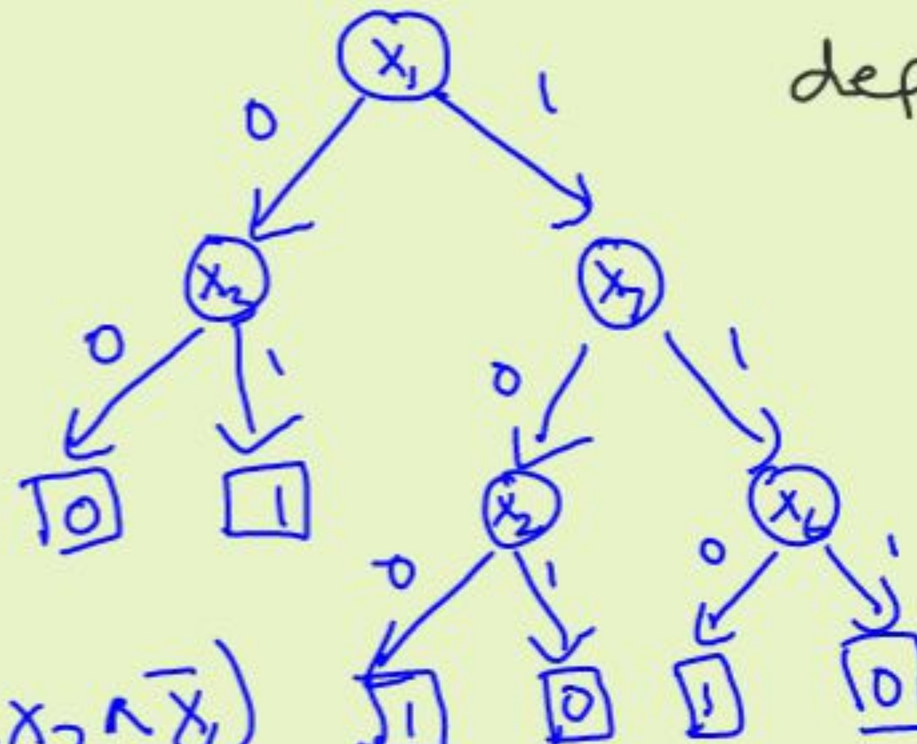

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Lemma: F a k-DNF or k-CNF.

[Håstad '86] $\Pr_{\rho \sim R_{q,n}} [\text{DT-depth}(F \restriction \rho) > \ell] \le O(qk)^{\ell}$.

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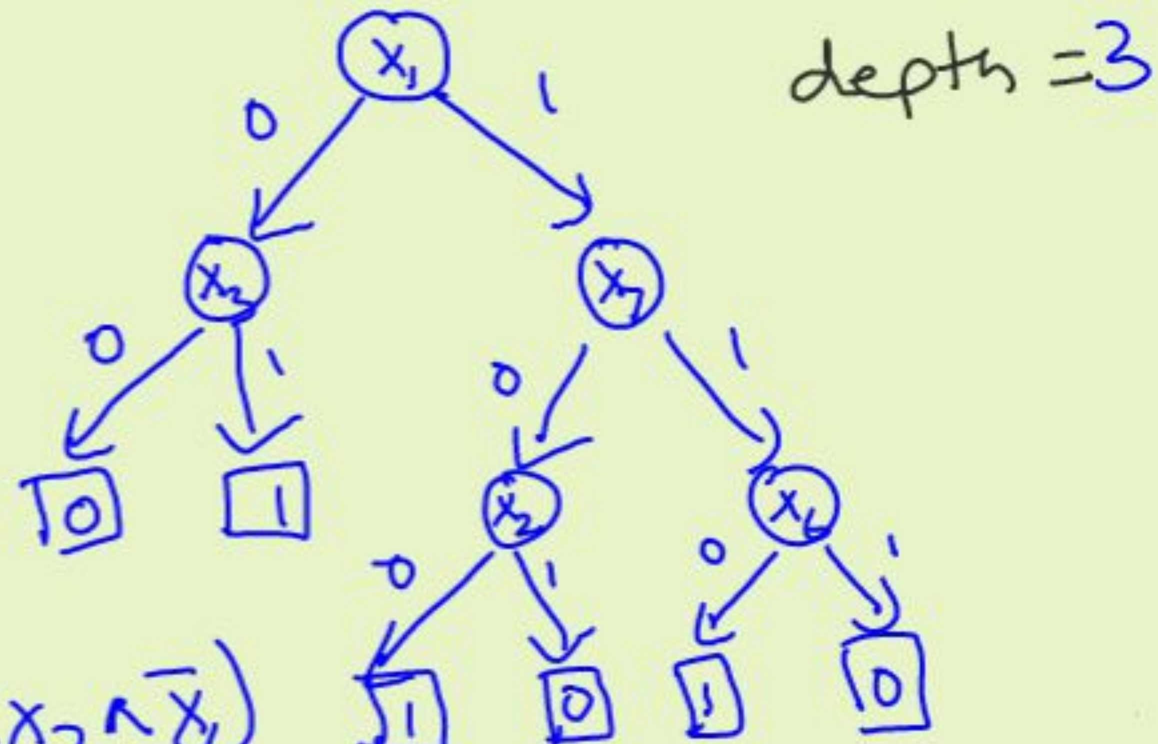

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Cor: F a k-DNF or k-CNF

$$\Pr_{\rho \sim R_{n/\alpha k}} [F \text{ not } \ell\text{-CNF or } \ell\text{-DNF}] \leq 2^{-\ell}.$$

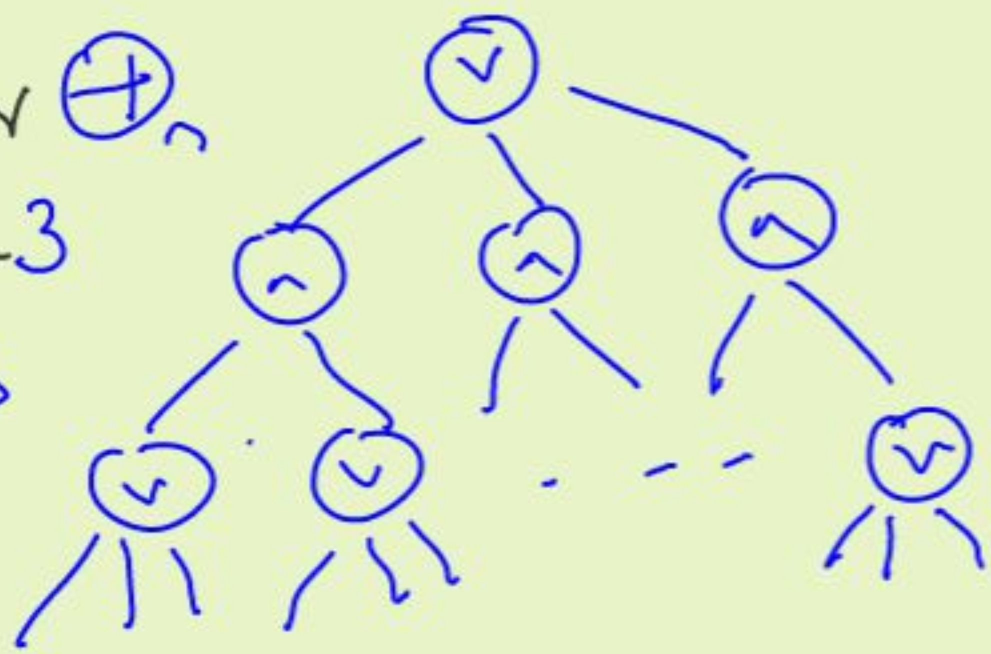
Switching lemma to lower bounds

Cor: F a k-DNF/CNF $\Pr_{\rho \sim R_{n/\alpha k}}[F \text{ not } \ell\text{-CNF/DNF}] \leq 2^{-\ell}$.

C for $\oplus_n$

depth-3

size s



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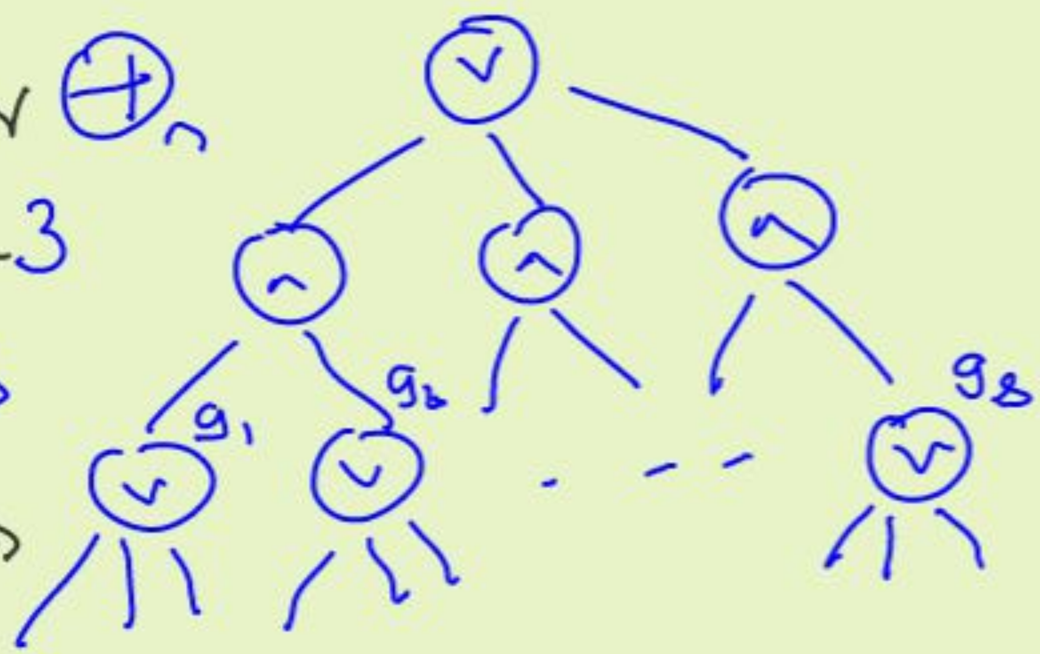
1-DNFs →



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$\ell = 2 \log s$

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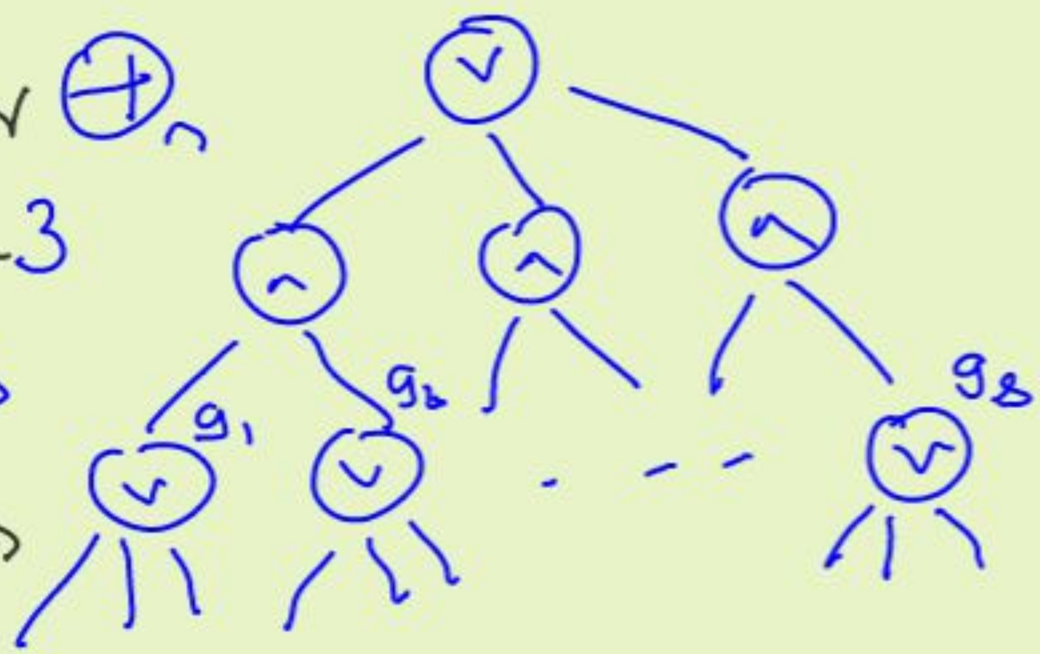
1-DNFs $\rightarrow$



$\Pr_{\rho_1 \sim R_{n/\alpha}}[g_i \text{ not } \ell\text{-CNF}] \le \frac{1}{s^2}$

$\Pr_{\rho_1 \sim R_{n/\alpha}}[\exists i : g_i \text{ not } \ell\text{-CNF}] \le \frac{1}{s}.$

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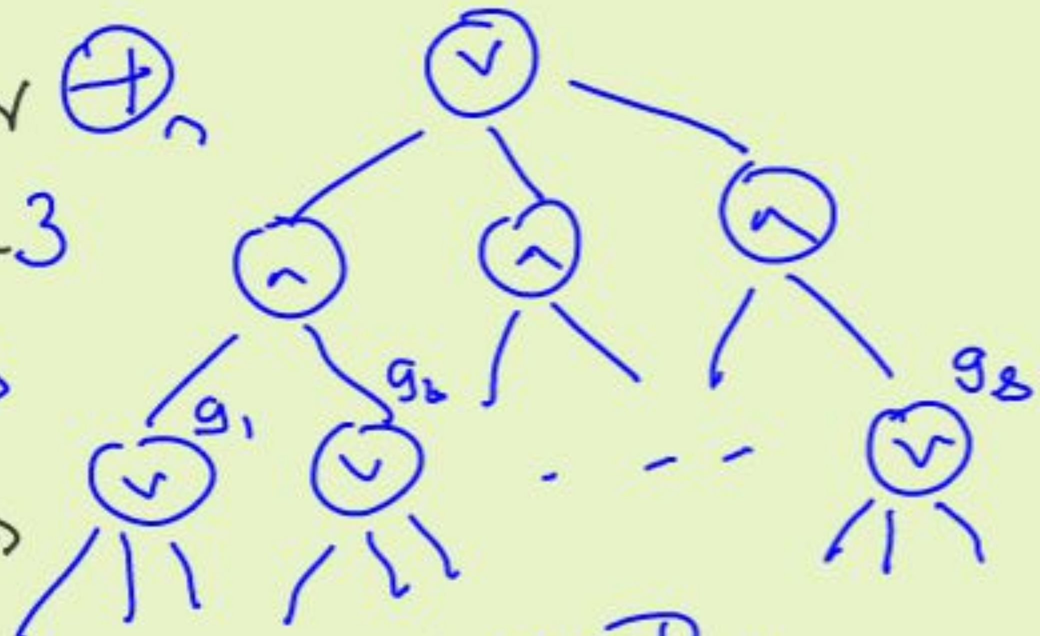

$\downarrow \rho_1 \sim R_{n/\alpha}$

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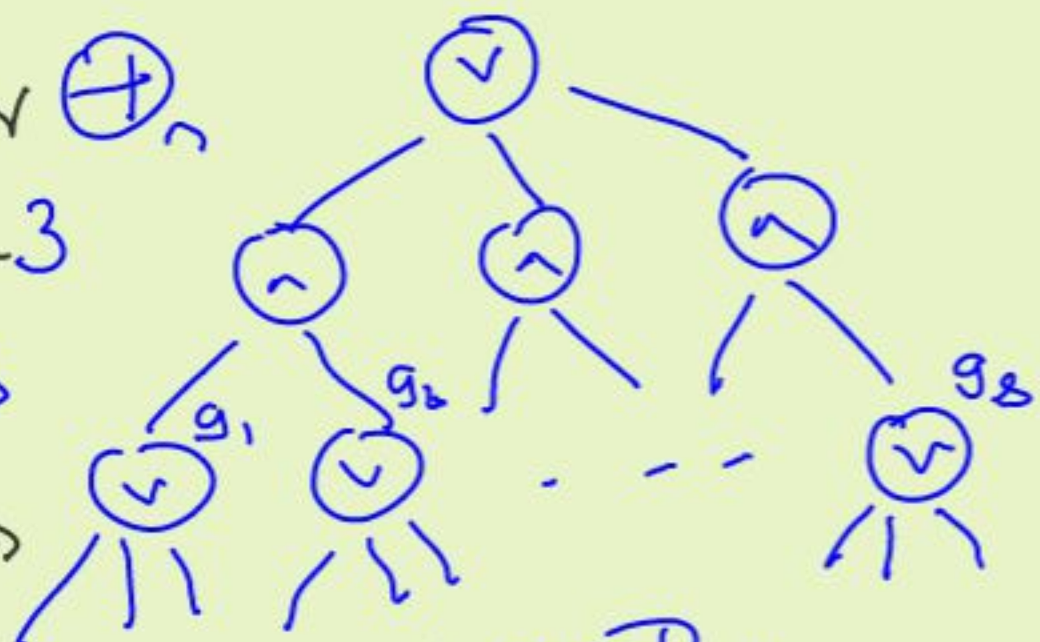

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$\downarrow \rho_1 \sim R_{n/\alpha}$

$\Pr_{\rho_2 \sim R_{n/\alpha \cdot \alpha \cdot \ell}}[\exists i : F_i \text{ not } \ell\text{-DNF}] \le \frac{1}{s}$

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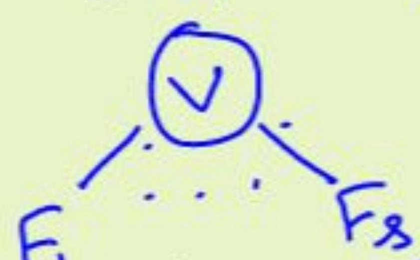

$\downarrow \rho_2 \sim R_{n/\alpha^2 \ell}$

C'' ℓ-DNF for $\oplus_{n/\alpha^2 \ell}$

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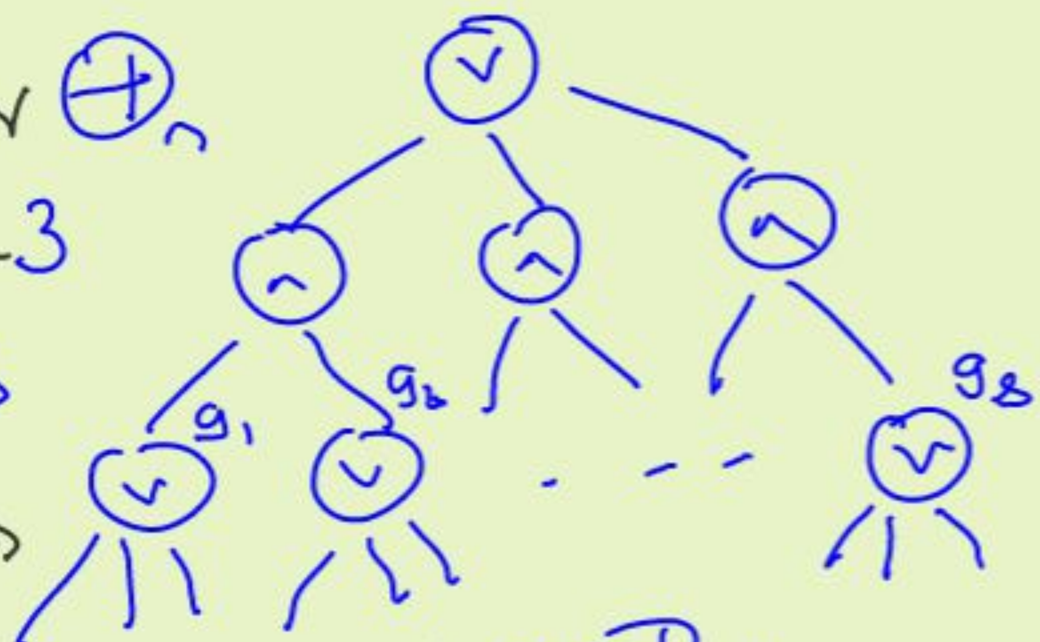

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ℓ-CNFs

$\downarrow \rho_2 \sim R_{n/\alpha^2\ell}$

C'' ℓ-DNF for $\oplus_{n/\alpha^2\ell}$

$\Rightarrow \quad \ell \ge n/\alpha^2\ell$

$\log s = \Omega(\ell) = \Omega(\sqrt{n})$.

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$$C_d \xrightarrow{\rho_1 \sim R_{n/\alpha}} C_d' \xrightarrow{\rho_2 \sim R_{n/\alpha^2 \ell}} \cdots C_{d-1} \xrightarrow{\rho_3 \sim R_{n/\alpha^3 \ell^2}} C_{d-2} \cdots \xrightarrow{\rho_{d-1} \sim R_{n/O(\ell)^{d-2}}} C_2$$

depth-d	depth-d	depth $d-1$	depth 2
size s	bottom fanin ℓ	...	bottom fanin ℓ
	size s at ht. ≥ 2	...	

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$$\ell \ge \frac{n}{O(\ell)^{d-2}} \Rightarrow \log s = \Omega(\ell) = \Omega(n^{1/d-1})$$

Baby Switching lemma

Lemma: F a k-DNF (ie width k) or k-CNF

$$\Pr_{\rho \sim R_{qn}} [F \upharpoonright \rho \text{ not constant}] \leq O(qk)$$

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no dependence on # terms/ clauses!

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$$\Pr_{\rho \sim R_{qn}} [F\restriction_\rho \text{ not constant}] \le O(qk)$$

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$$BAD = \{\rho : F\restriction_\rho \text{ not constant}\} \qquad |BAD| \le O(qk) \cdot R_{qn}$$

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Pf: 1-1 map $\pi : BAD \to R_{qn-1} \times [k]$

$$\longrightarrow |R_{qn-1}| \leq O(q) \cdot |R_{qn}|$$

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$$F\restriction_\rho = \vee$$

$$T_1\restriction_\rho \;\ldots\; T_i\restriction_\rho \;\ldots\; T_s\restriction_\rho$$

$$\neq 1 \qquad \neq 1 \qquad \neq 1$$

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$$\Pr_{\rho \sim R_{an}} [F\restriction\rho \text{ not constant}] \le O(ak)$$

$$\Updownarrow$$

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Pf: 1-1 map $\pi : BAD \to R_{an-1} \times [k]$

$$|R_{an-1}| \le O(a)\cdot|R_{an}|$$

$$F\restriction\rho = \vee(T_1\restriction\rho \ldots T_i\restriction\rho \ldots T_s\restriction\rho)$$

$\not\equiv 1 \quad \not\equiv 1 \quad \not\equiv 1$

ρ': → Find $i^* = \min\{i \mid T_i\restriction\rho \not\equiv 0\}$

→ Set first unset literal L to 1.

$\pi(\rho) = (\rho', j)$ — j: Index of L in T_{i^*}.

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Pf: 1-1 map $\pi : BAD \to R_{qn-1} \times [k]$

$\to |R_{qn-1}| \le O(q)\cdot |R_{qn}|$

$F\restriction_\rho = \vee(T_1\restriction_\rho, \dots, T_i\restriction_\rho, \dots, T_s\restriction_\rho)$

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ρ': $\to$ Find $i^* = \min\{i \mid T_i\restriction_\rho \not\equiv 0\}$

$\to$ Set first unset literal L to 1.

$\pi(\rho) = (\rho', j)$ — $j \to$ Index of L in T_{i^*}.

Decode (ρ', j): $i^* = \min\{i : T_i\restriction_{\rho'} \not\equiv 0\}$

↳ Unset jth literal get ρ.

Proof of Switching lemma [Razborov '93, Beame '94]

Lemma: F a k-DNF/CNF. $\Pr_{\rho \sim R_{q,n}} [\text{DT-depth}(F\restriction\rho) \geq l] \leq O(qk)^{l}$.

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Pf: $F \upharpoonright \rho$

$$\vee$$

$$T_1' = T_1 \upharpoonright \rho \quad \cdots \quad T_s \upharpoonright \rho$$

$\text{BAD} = \{\rho \mid \text{DT-depth}(F \upharpoonright \rho) \geq \ell\}$

$\pi : \text{BAD} \to R_{q,n-\ell} \times [k]^\ell \times \{0,1\}^\ell$

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Pf: $F\restriction\rho$

$$T_1' = T_1\restriction\rho \longrightarrow \vee \longleftarrow T_s\restriction\rho \qquad (T_1' = T_1\restriction\rho, \ \dots, \ T_s\restriction\rho)$$

$BAD = \{\rho \mid \text{DT-depth}(F\restriction\rho)\geq \ell\}$

$\pi : BAD \rightarrow R_{q,n-\ell} \times [k]^{\ell} \times \{0,1\}^{\ell}$

Canonical DT

→ If all $T_i' \equiv 0$, o/p 0

→ else

$i^* = \min\{i \mid T_i' \not\equiv 0\}$

Query all vars. in T_{i^*}'

& o/p 1 if satisfied

→ Substitute in F & repeat.

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Pf: $F\restriction\rho$ $\qquad T_1' = T_1\restriction\rho \;\rightarrow\; \vee \;\leftarrow\; T_s\restriction\rho$ $\quad \cdots$

$\text{BAD} = \{\rho \mid \text{DT-depth}(F\restriction\rho) \geq \ell\}$

$\pi : \text{BAD} \rightarrow R_{qn-\ell} \times [k]^{\ell} \times \{0,1\}^{2\ell}$

Canonical DT

→ If all $T_i' \equiv 0$, o/p 0

→ else
$i^* = \min\{i \mid T_i' \not\equiv 0\}$

Query all vars. in T_{i^*}'
& o/p 1 if satisfied

→ Substitute in F & repeat.

SL-Encode$(\rho) = (\rho', u, v)$

→ $\rho' = \rho$

→ At each term T_{i^*}' queried by CDT, set literals to 1 in ρ'

→ $u \in [k]^{\ell} \times \{0,1\}^{\ell}$ – literals queried & "first bit" indicator

→ $v \in \{0,1\}^{\ell}$ – original assignment

$\text{CDT}(F\restriction\rho)$

Proof of Switching Lemma [Razborov '93, Beame '94]

Lemma: F a k-DNF/CNF. $\Pr_{\rho \sim R_{q,n}} [\text{DT-depth}(F\restriction\rho) \geq \ell] \leq O(qk)^{\ell}$.

Pf: $F\restriction\rho$ $\quad \vee$

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SL-Decode(ρ', u, v)

→ $i^* = \min\{i : T_i\restriction\rho' \not\equiv 0\}$

→ Use u to find queried variables & v to reset.

→ Repeat.

Multi-Switching Lemma [Håstad'14]

→ Switching lemma for systems of DNFs / CNFs.

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→ $F_1, \ldots, F_s$ k-DNFs

→ $\Pr_{\rho \sim R_{n/\alpha k}} \left[\exists i : F_i \restriction \rho \text{ not } \ell\text{-CNF} \right] \leq s \cdot 2^{-\ell}$

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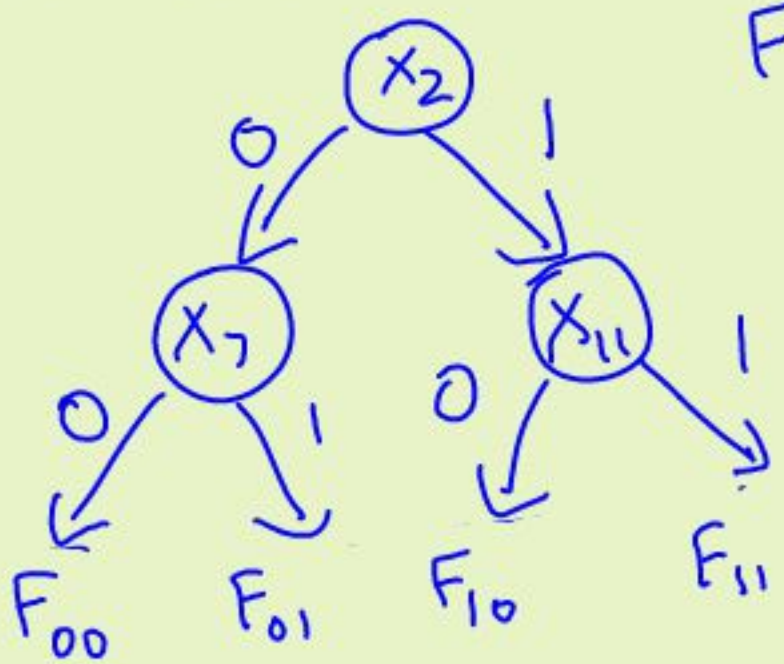
F a DNF/CNF



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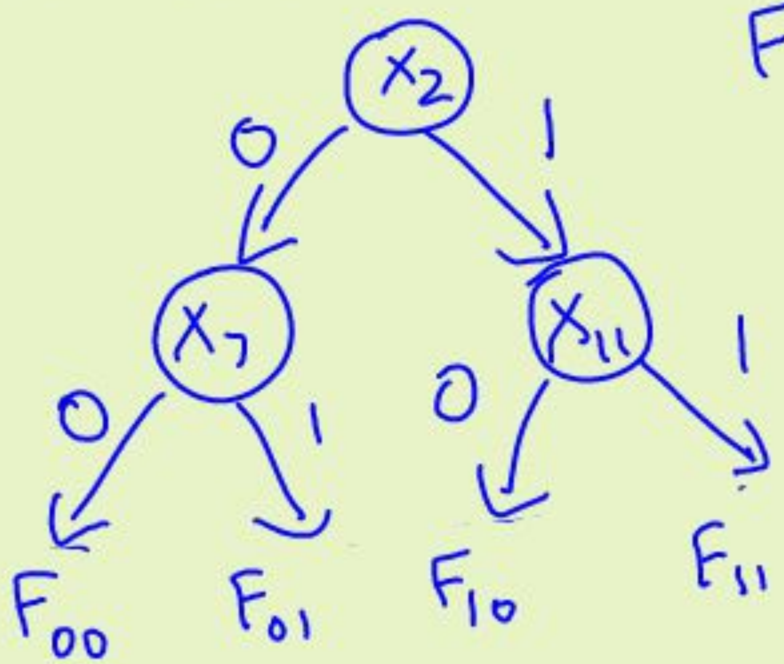

__Want:__ RT s.t. $(F_i \restriction \rho) \restriction \lambda$ a depth-ℓ DT $\forall i$ $\forall$ leaves λ

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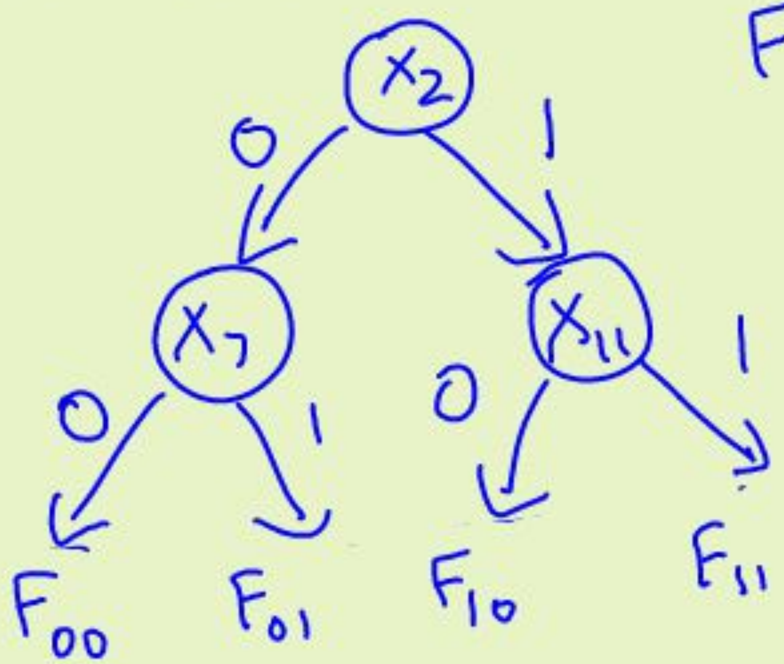

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RT ℓ-depth$(F_1, \dots, F_s)$

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→ Restriction Tree (RT)

Multi-Switching Lemma $F_1, \dots, F_s$ k-DNFs / CNFs, $\ell \geq \log s$

$$\Pr_{\rho \sim R_{2n}} [\ RT_\ell\text{-depth}(F_1, \dots, F_s) \geq D] \leq O(q, k)^D.$$

Proof of Multi-Switching lemma [Håstad'14, Servedio-Tan'19]

MSL: $F_1, \ldots, F_s$, $\ell \geq \log s$ $\quad \Pr_{\rho \sim R_{qn}} \left[RT_\ell\text{-depth}(F_1, \ldots, F_s) \geq D \right] \leq O(qk)^D$.

k-DNFs / CNFs

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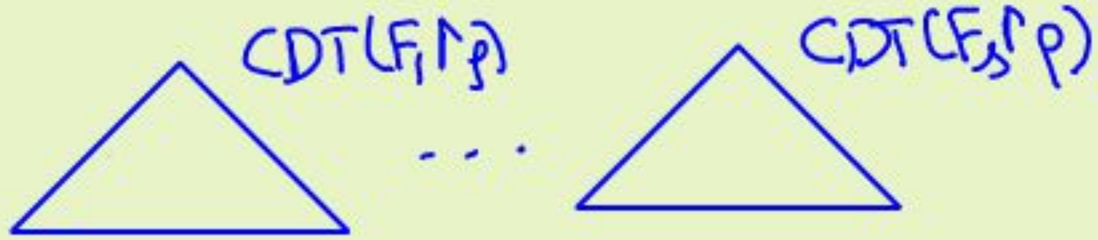

$BAD = \{\rho \mid RT_\ell\text{-depth}(F_1, \dots, F_s) \geq D\}$

$\pi : BAD \rightarrow R_{qn-D} \times \text{Advice}$

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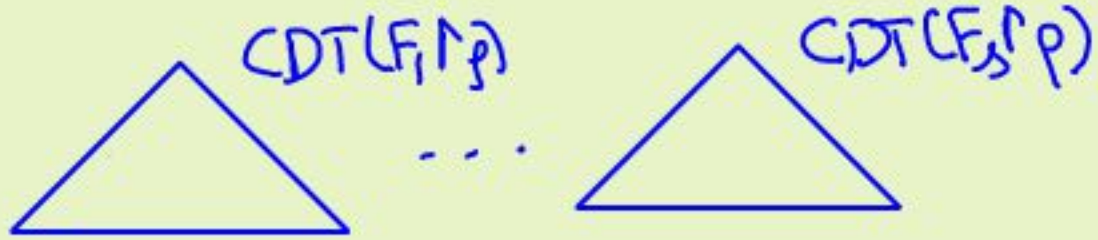


$BAD = \{\rho \mid RT_\ell\text{-depth}(F_1, \ldots, F_s) \geq D\}$

$\pi : BAD \to R_{qn-D} \times$ Advice

Canonical RT

→ If $\text{depth}(CDT(F_i\restriction\rho)) \leq \ell$ $\forall i$, stop.

→ else $i^* = \min\{i \mid \text{depth} > \ell\}$

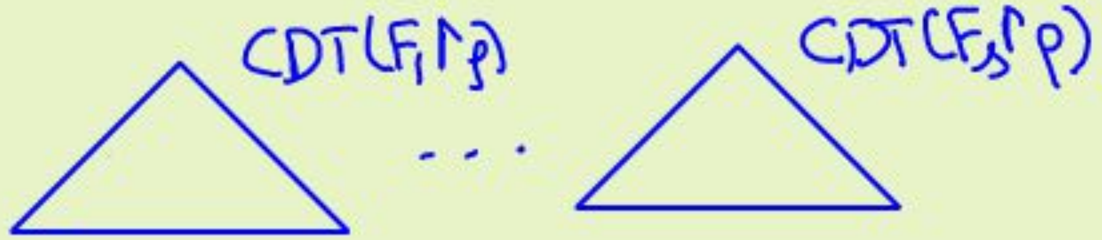
→ Find left-most path τ of length $> \ell$ & query all vars.

→ Substitute & repeat.

Proof of Multi-Switching lemma [Håstad'14, Servedio-Tan'19]

MSL: $F_1, \dots, F_s$, $\ell \geq \log s$ $\quad \Pr\left[RT_\ell\text{-depth}(F_1, \dots, F_s) \geq D\right] \leq O(qk)^D$.

k-DNFs / CNFs $\quad \rho \sim R_{qn}$



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MSL-Encode $(\rho) = (\rho', a_0, a_1)$

→ $\rho' = \rho$, $a_0 = a_1 =$ empty

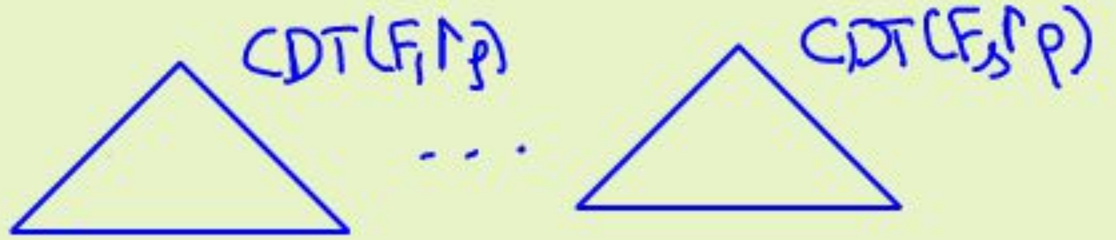
→ At each path τ queried by CRT, use SL-encoding to augment ρ', a_0

→ $a_1 = a_1 \circ (i^*$, CRT assignment to τ, firstbit indicator$)$

Proof of Multi-Switching lemma [Håstad'14, Servedio-Tan'19]

MSL: $F_1, \dots, F_s$, $\ell \geq \log s$ $\quad \Pr_{\rho \sim R_{an}} \left[RT_\ell\text{-depth}(F_1, \dots, F_s) \geq D \right] \leq O(qk)^D$.

k-DNFs / CNFs



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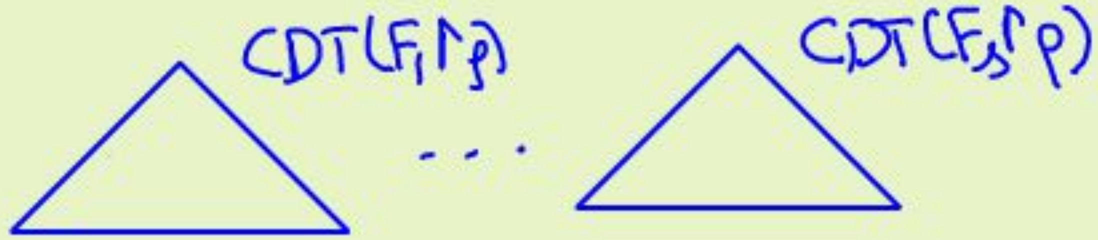
Proof of Multi-Switching lemma [Håstad'14, Servedio-Tan'19]

MSL: $F_1, \ldots, F_s$, $\ell \geq \log s$ $\Pr_{\rho \sim R_{\alpha n}}\left[RT_\ell\text{-depth}(F_1, \ldots, F_s) \geq D\right] \leq O(\alpha k)^D$.

k-DNFs / CNFs



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$\to$ If $\text{depth}(CDT(F_i \restriction \rho)) \leq \ell$ $\forall i$, stop.

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$\left|[s]^{D/\ell} \times \{0,1\}^{2D}\right| = O(1)^D$

Proof of Multi-Switching lemma [Håstad'14, Servedio-Tan'19]

MSL: $F_1, \ldots, F_s$, $\ell \geq \log s$ k-DNFs / CNFs

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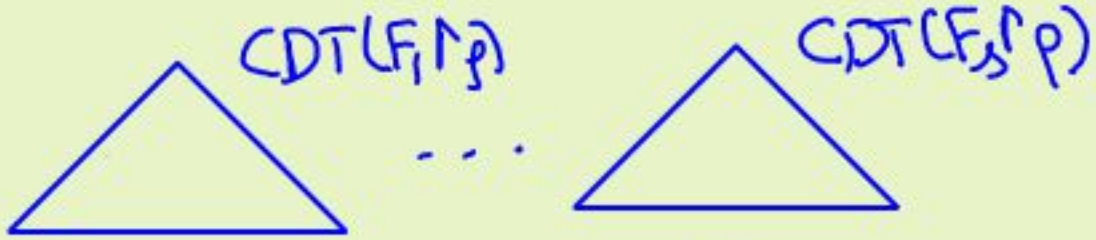

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MSL-Decode (ρ', a_0, a_1)

→ Use a_1 to find first i^* & # vars read in first τ

→ Use SL-Decode to find vars & path in CRT.

→ Set & continue.

MSL-Encode $(\rho) = (\rho', a_0, a_1)$

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→ At each path τ queried by CRT, use SL-encoding to augment ρ', a_0.

→ $a_1 = a_1 \circ (\text{length}(\tau), i^*)$