Geometry of tropical varieties

Lecture 2

28/06/2022

Summary

k: base field

K: valued field, $K \supseteq k$, $K = \overline{K}$

$\quad \nu : K \to \mathbb{R} \cup \{\infty\}$

For $\underline{x} = (x_1, \dots, x_n) \in K^{\times n}$, $\quad \operatorname{trop}(\underline{x}) = (\nu(x_1), \dots, \nu(x_n))$

$X(K) \subseteq (K^{\times})^n \qquad \operatorname{Trop}(X) = \overline{\{\operatorname{trop}(\underline{x})\}} \subseteq \mathbb{R}^n$

$X \hookrightarrow T_k^n$

$I \subseteq k[T^n] = k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ $\qquad$ Fundamental theorem $\quad \operatorname{Trop}(X) = \bigcap_{f \in I}$ "zero set" of $\operatorname{trop}(f)$

$f = \Sigma\, c_J \underline{x}^J \qquad \operatorname{trop}(f) = \bigoplus_J \nu(c_J) \odot \underline{x}^{\odot J}$, $\quad$ "zero set" $= \left\{ \underline{a} = (a_1, \dots, a_n) \;\middle|\; \min\{\nu(c_J) + \langle \underline{a}, J \rangle\} \text{ achieved twice} \right\}$

Example : Complement of hyperplane arrangements

$H_0, \dots, H_m \subseteq \mathbb{P}^n_{\mathbb{C}}$ hyperplanes $\qquad$ H_j zero set of linear form

$$l_j = \sum_{k=0}^{n} b_k z_k$$

$$X = \mathbb{P}^n_{\mathbb{C}} \setminus \bigcup_{j=0}^{m} H_j$$

We get a map

$$\begin{array}{ccc} X & \longrightarrow & T^{m+1}/T^1 \simeq T^m \\ \underline{x} & \longrightarrow & (l_0(x), \dots, l_m(x)) \\ \Uparrow & & \Uparrow \\ X(\mathbb{C}) & & \mathbb{C}^{\times(m+1)}/\mathbb{C}^\times \end{array}$$

This gives an embedding $X \hookrightarrow T_{\mathbb{C}}^{m+1}/T_{\mathbb{C}}^1 \simeq T_{\mathbb{C}}^m$ provided that $l_0, \dots, l_m$ generate the whole space of linear forms.

Equations for X

For $A \subseteq \{0, 1, \dots, m\}$ dependent, that is, such that

$$\sum_{j \in A} c_j l_j = 0 \quad \text{for} \quad c_j \in \mathbb{C}$$

we get an equation $\sum_{j \in A} c_j x_j$ for $X \hookrightarrow T^{m+1}/T^1$

$(x_0, \dots, x_m)$ coordinates in T^{m+1}

Theorem (Ardila-Klivans)

$$\mathrm{Trop}(X) = \bigcap \text{“zero set” of } \bigoplus_{j \in A} x_j \quad \text{for } A \text{ dependent}$$

$$= \left\{ (a_0, \dots, a_m) \in \mathbb{R}^{m+1}/\mathbb{R}(1, \dots, 1) \;\middle|\; \min_{j \in A} \{a_j\} \text{ is achieved at least twice} \right\}$$

for all A dependent

Hyperplane arrangement $H_0, \dots, H_m$ defines a matroid M on the ground set $E = \{0, \dots, m\}$.

Bases $B \subseteq \{0, \dots, m\}$ $\{l_i\}_{i \in B}$ form a basis

Independent sets

Circuits

Flats (closed sets)

$$\mathrm{Trop}(X) = \left\{ (a_0, \dots, a_m) \in \mathbb{R}^{m+1}/\mathbb{R}(1, \dots, 1) \;\middle|\; \text{for all } A \text{ dependent, } \min_{i \in A} a_i \text{ is achieved twice} \right\}$$

$$\underline{a} = \alpha_1 \mathbb{1}_{F_1} + \alpha_2 \mathbb{1}_{F_2} + \dots + \alpha_k \mathbb{1}_{F_k}$$

$$\emptyset \neq F_1 \subsetneq F_2 \subsetneq \dots \subsetneq F_k \neq E$$

$$\alpha_1, \dots, \alpha_k > 0$$

Proposition • $F_1, \dots, F_k$ are closed in M.

$\mathcal{F}: \quad \emptyset \neq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k \neq E$ flag of flats (closed sets)

$\sigma_{\mathcal{F}} := \mathbb{R}_+ 1_{F_1} + \cdots + \mathbb{R}_+ 1_{F_k}$

Cone generated by $1_{F_1}, \dots, 1_{F_k}$

$$\mathrm{Trop}(X) = \bigcup_{\mathcal{F}} \sigma_{\mathcal{F}}$$

this gives to Trop(X) the structure of a rational fan.

<u>Definition</u> (Bergman fan of a matroid)

Let M be a matroid on the ground set $E = \{0, 1, \dots, m\}$.

The Bergman fan of M denoted Σ_M is the union of cones $\sigma_{\mathcal{F}} \subseteq \mathbb{R}^{m+1}/\mathbb{R}(1,\dots,1)$

for $\mathcal{F}: \emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_k \subsetneq E$, $k \in \mathbb{N}$ a flag of flats in M and

$\sigma_{\mathcal{F}}$: convex cone generated by $1_{F_1}, \dots, 1_{F_k}$, $\quad \sigma_{\mathcal{F}} = \mathbb{R}_+ 1_{F_1} + \cdots + \mathbb{R}_+ 1_{F_k}$.

§ 5 Properties of tropicalizations

k base field $\qquad X \subset T_k^n$

$K \supseteq k$ valued field $\qquad I \subseteq k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ ideal of definition

$v : K \to \mathbb{R} \cup \{\infty\}$

(1) $v|_k$ trivial $\quad (k = \mathbb{C} , \ K = \overline{\mathbb{C}((t))})$

Fondamental theorem implies $\quad \mathrm{Trop}(X) = \{ (a_1, \dots, a_n) \in \mathbb{R}^n \mid \forall f = \sum c_J x^J \in I$
$\min_J \{ \underbrace{v(c_J)}_{\substack{=0 \\ \text{if } c_J \neq 0}} + a_1 j_1 + \dots + a_n j_n \}$ achieved twice $\}$

$\Rightarrow$ $\mathrm{Trop}(X)$ is invariant under multiplication by scalars in $\mathbb{R}_+$

$(a_1, \dots, a_n) \in \mathrm{Trop}(X) \Rightarrow \forall \lambda \in \mathbb{R}_+ \quad (\lambda a_1, \dots, \lambda a_n) \in \mathrm{Trop}(X).$

Theorem (Finiteness Property) $\quad \exists f_1, \dots, f_N \in I$ such that $\mathrm{Trop}(X) = \bigcap_{j=1}^{N}$ "zero set" of $\mathrm{trop}(f_j)$

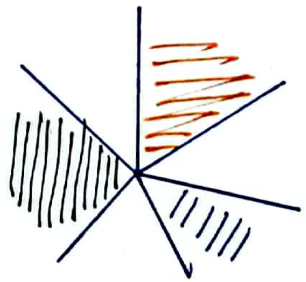
theorem (Polyhedral property) $\quad \mathrm{Trop}(X)$ is the support of a rational fan in $\mathbb{R}^n$.

Theorem (Dimension) $\quad \mathrm{Trop}(X)$ has dimension $=$ dim X

Definition (Fanfold ; tropical fanfold)

- A subset $Y \subseteq \mathbb{R}^n$ is called a fanfold if it is support of a rational fan in $\mathbb{R}^n$. That is, $\exists$ a collection $\Sigma = \{\sigma\}$ consisting of rational cones $\sigma \subseteq \mathbb{R}^n$ such that
  - $\sigma \in \Sigma$, τ face of $\sigma \Rightarrow \tau \in \Sigma$
  - $\sigma, \eta \in \Sigma \Rightarrow \sigma \cap \eta$ common face of σ and η.

  and $Y = \bigcup_{\sigma \in \Sigma} \sigma$.



- A tropical fanfold is a fanfold endowed with a weight function ω which verifies the BALANCING CONDITION.

- A reduced tropical fanfold is a tropical fanfold in which $\omega \equiv 1$.

Theorem (Structure theorem : local case)

$X \hookrightarrow T^n_K$ $\qquad$ X irreducible (I prime)

I ideal of definition $\qquad$ d = dimension of X

- Trop(X) is pure dimensional = dimension of X.
- It is endowed with a weight function ω defined on d-dimensional cones Σ_d of a fan structure on Trop(X).

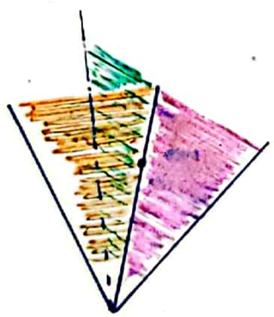
$$\omega : \Sigma_d \to \mathbb{N}$$
$$\sigma \mapsto \omega(\sigma)$$

- The following balancing property holds

For all cone $\tau \in \Sigma$ of dimension $d-1$

$$\sum_{\sigma \supsetneq \tau} \omega(\sigma)\, e_{\sigma/\tau} = 0 \quad \text{in } N/N_\tau .$$

That is, $\sum_{\sigma \supsetneq \tau} \omega(\sigma)\, e_{\sigma/\tau}$ tangent to τ.



(2) Structure theorem : general case

$X \subset T^n_K$ irreducible

k base field

K valued field

$k \subseteq K$ $\quad K = k$ allowed

- $\mathrm{Trop}(X)$ is the support of a rational polyhedral structure Σ

$\Sigma = \{\sigma\}$, σ rational polyhedra

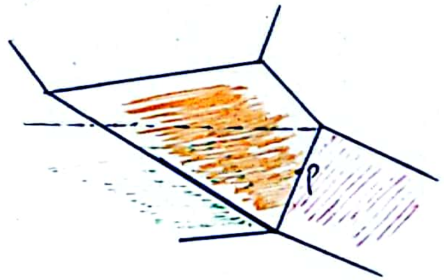
- $\sigma \in \Sigma$, τ face of σ $\Rightarrow$ $\tau \in \Sigma$
- $\sigma \cap \eta$ non-empty $\Rightarrow$ $\sigma \cap \eta$ common face of σ, η.

- It is pure dimensional $= \dim(X) = d$

- Endowed with a weight function ω such that the balancing property holds

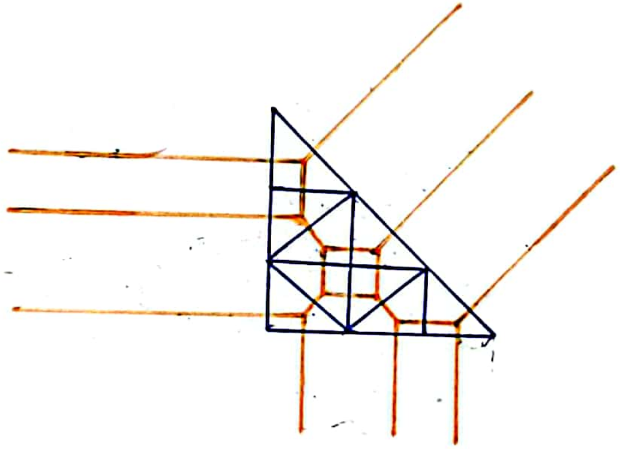
$$\forall \tau \in \Sigma_{d-1}, \qquad \sum_{\sigma \supsetneq \tau} \omega(\sigma)\, e_{\sigma/\tau} = 0 \text{ in } N_{\sigma/\tau}.$$

- Locally, at each point $x \in \mathrm{Trop}(X)$ we get a tropical fanfold.
This tropical fanfold is the tropicalization of some subvarieties of T^n



Local fanfold at p

Example : Hypersurfaces



Example Bergman fanfold

M matroid on $E = \{0, 1, \dots m\}$

Σ_M is a reduced tropical fanfold.

Top dimensional cones

$\emptyset \neq F_1 \subsetneq \cdots \subsetneq F_r \neq E$

maximal chains

Codimension one cones

$\emptyset \neq F_1 \subsetneq \cdots \subsetneq F_{r-1} \neq E$

Balancing condition holds.