MONOTONE CIRCUIT LOWER BOUNDS VIA THE APPROXIMATION METHOD

SRIKANTH SRINIVASAN

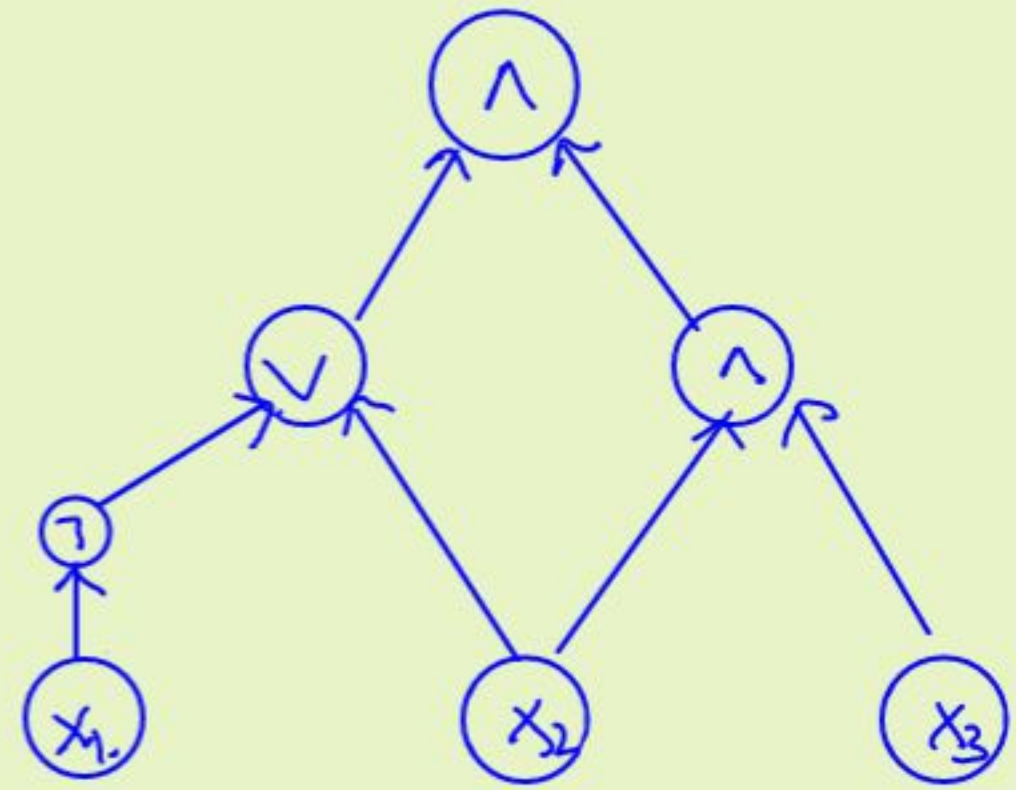
UNIV. OF COPENHAGEN

Circuit Complexity

→ Boolean circuit

De Morgan basis

AND / OR / NOT



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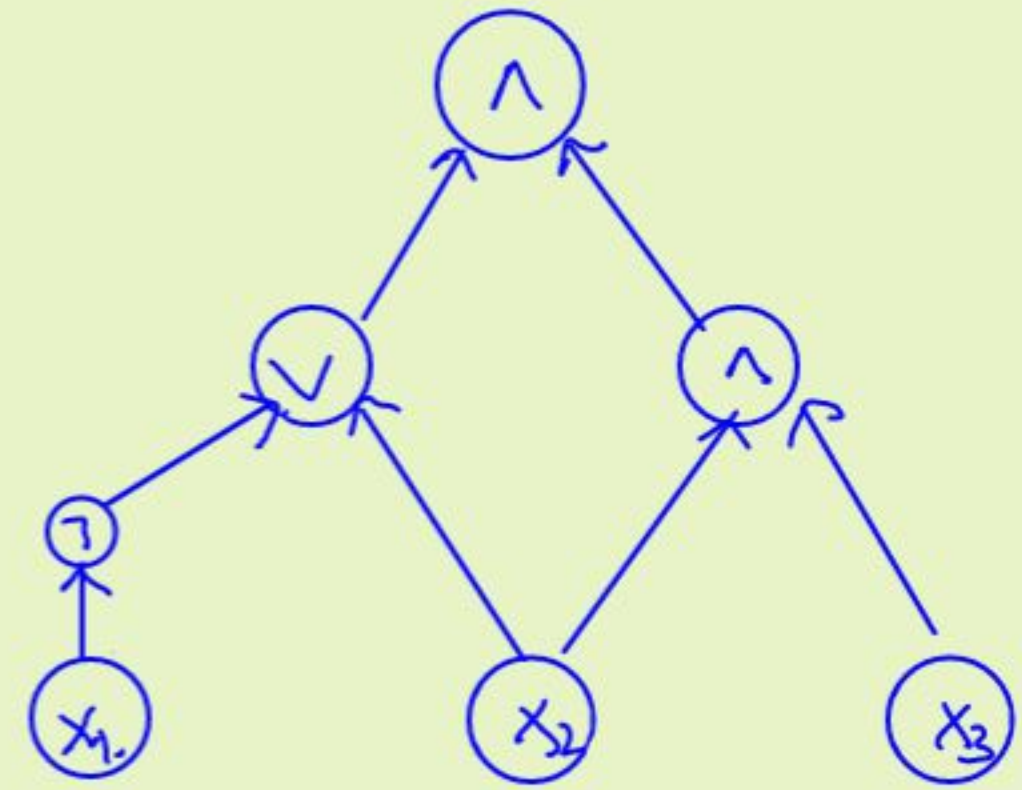

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Depth = longest AND/OR path = 2.

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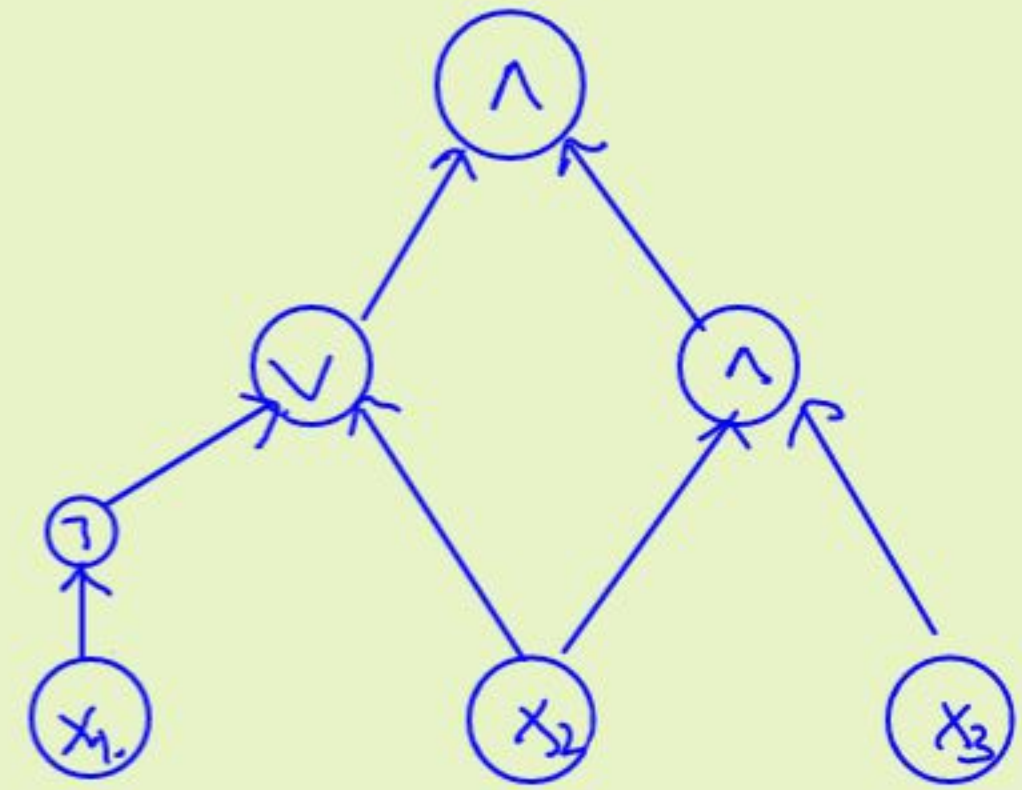

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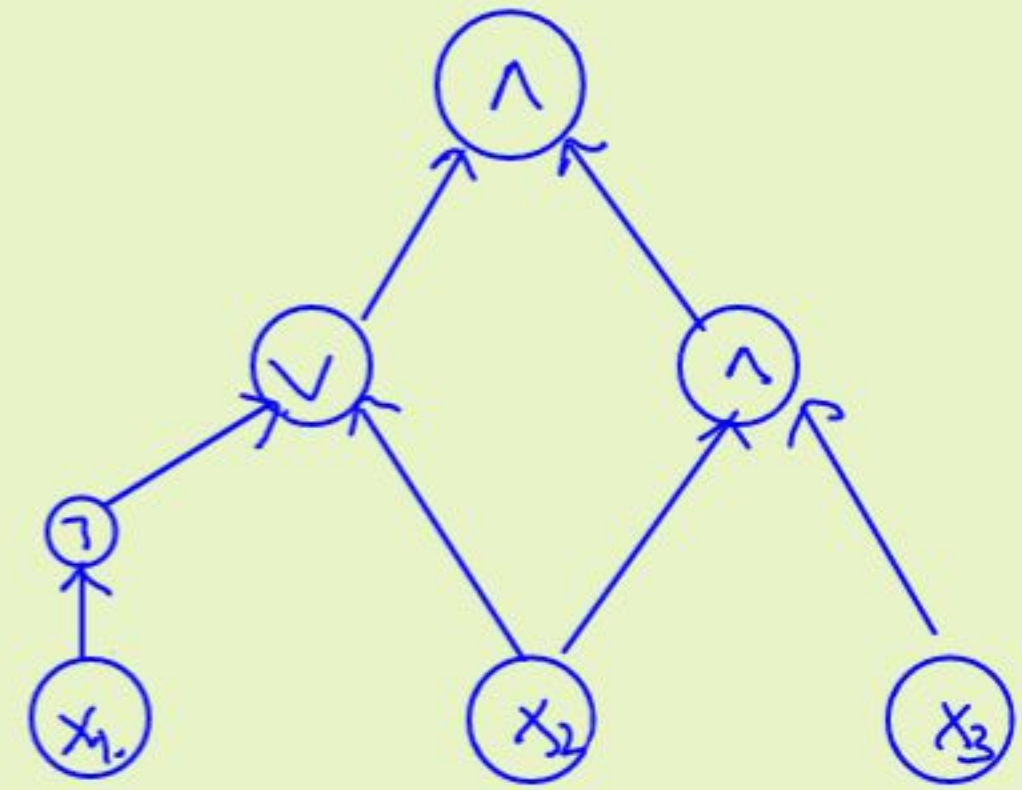
→ Any f has a circuit of size $O(2^n)$.

→ Most f need size $2^{\Omega(n)}$

→ Explicit f ?



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$\swarrow$

defines a graph

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Conj: $\text{Clique}_{n,\sqrt{n}}$ has no circuits of size $\text{poly}(n)$.

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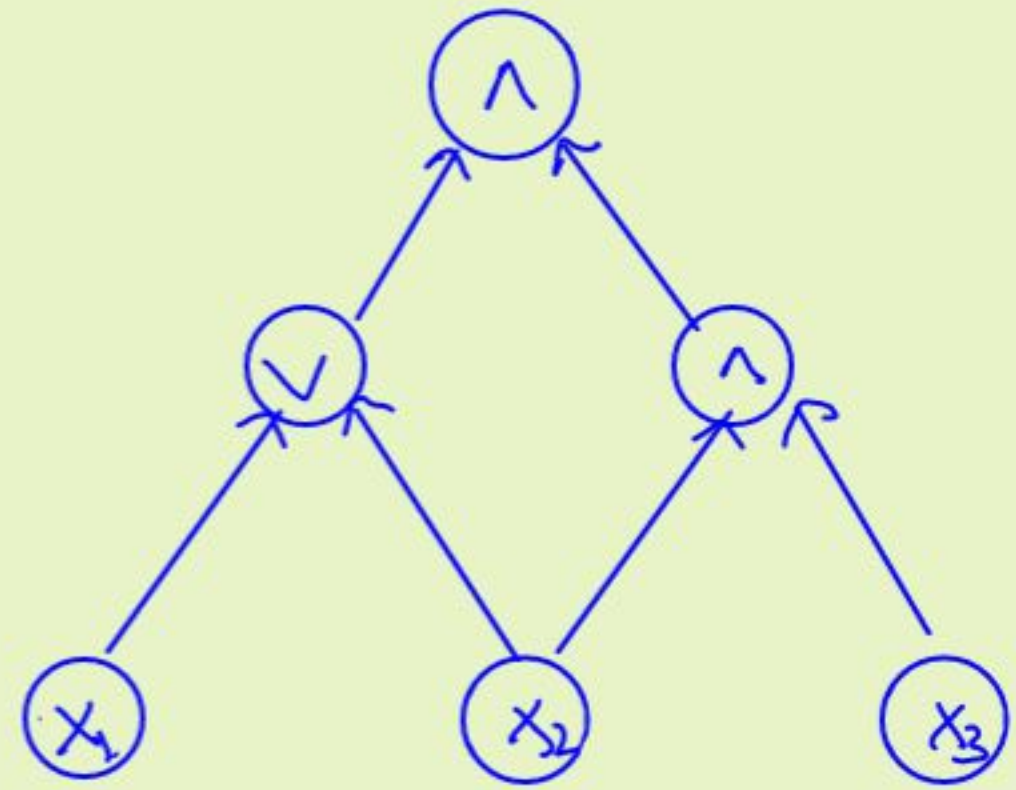

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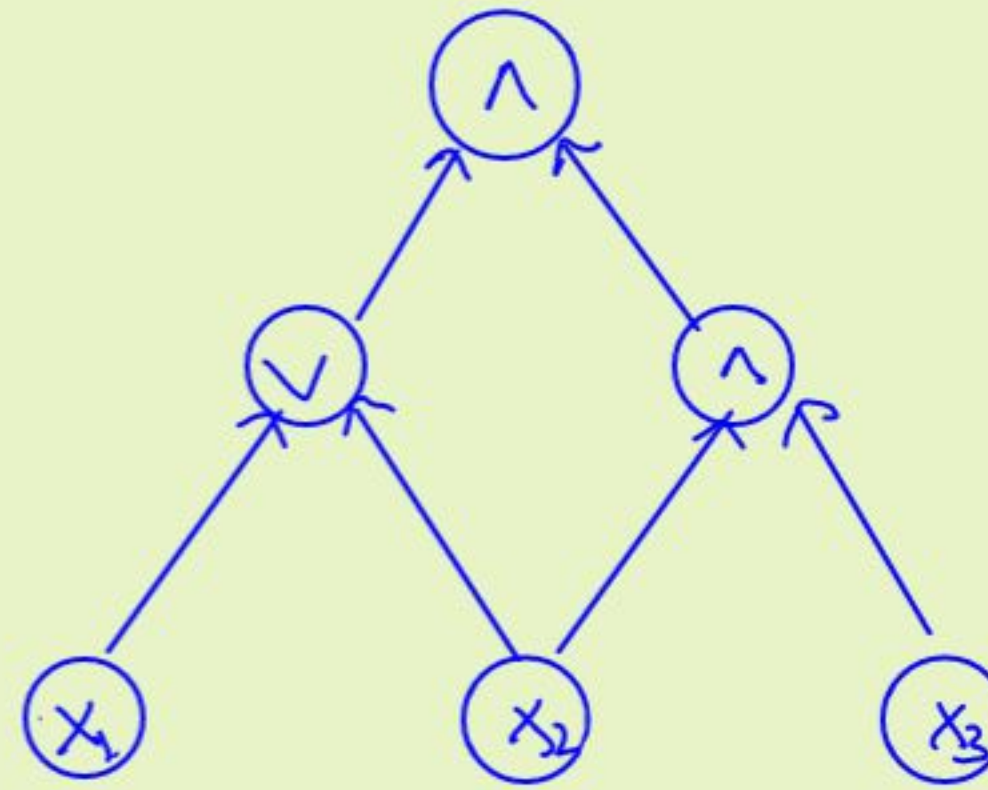

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→ Explicit f? $\text{Clique}_{n,k}$?

Main theorem [Razborov '85, Alon-Boppana '87]

Say $k \leq n^{2/3-\varepsilon}$. Any monotone circuit for $\text{Clique}_{n,k}$ has size $n^{\Omega(\sqrt{k})}$.

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→ Today: Above theorem for $k = n^{\Omega(1)}$
[Proof works for $k \le n^{1/2-\varepsilon}$]

Baby case: $k=3$ (triangles)

Thm: [Razborov'85] Any monotone circuit for $\mathrm{Clique}_{n,3}$ has size $n^3/\mathrm{polylog}(n)$.

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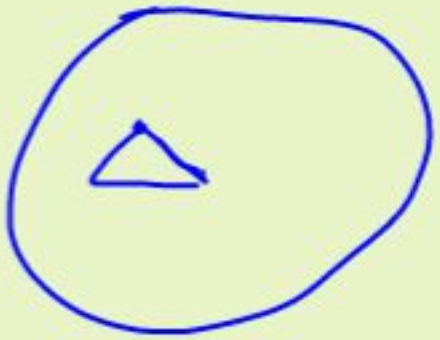

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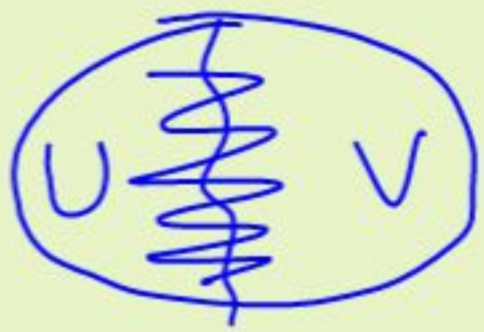
Hard 0 inputs: Graphs with many edges



Complete bipartite graph

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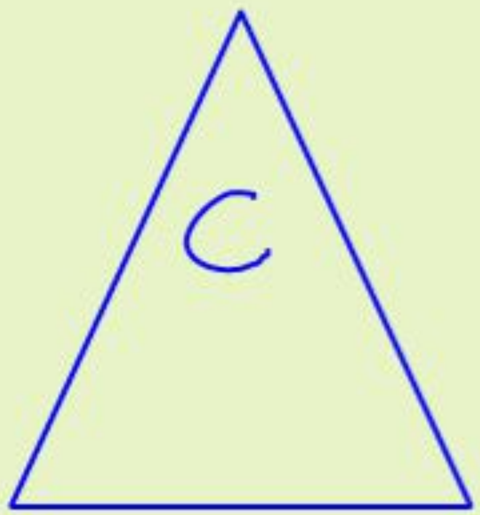

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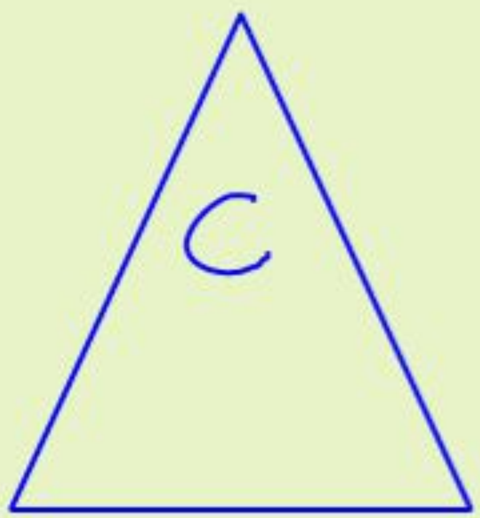

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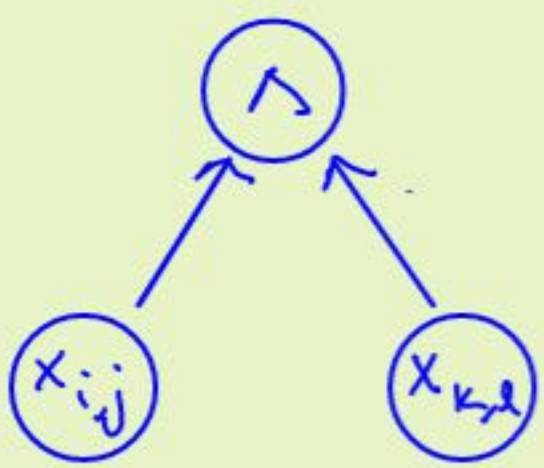

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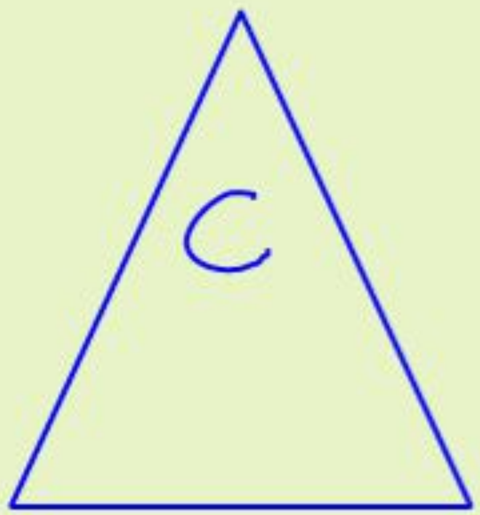

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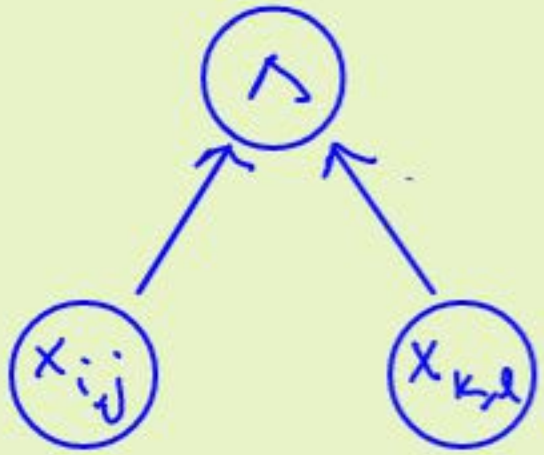

$$\Pr_{D_1}[x_{i,j} \wedge x_{k,\ell} = 1] \leq O(1/n^3)$$

Obs: Replacing s such gates by 0 creates negligible error w.r.t. D_1

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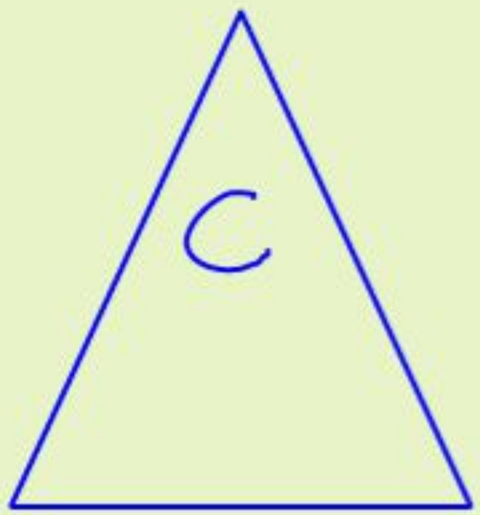

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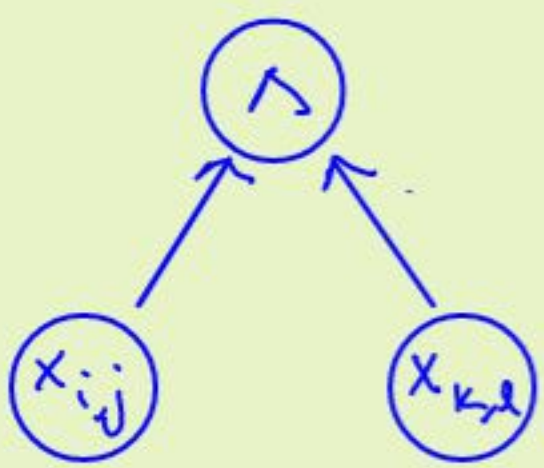

$$\Pr_{D_1}[x_{i,j} \wedge x_{k,l} = 1] \leq O(1/n^3)$$

Obs: Replacing s such gates by 0 creates negligible error w.r.t. D_1 & no error w.r.t. D_0 [monotonicity!]

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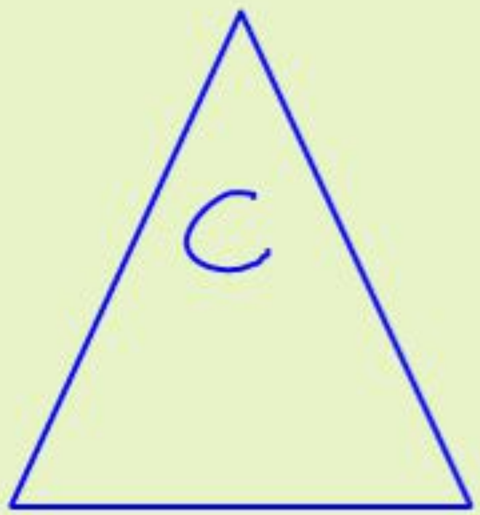

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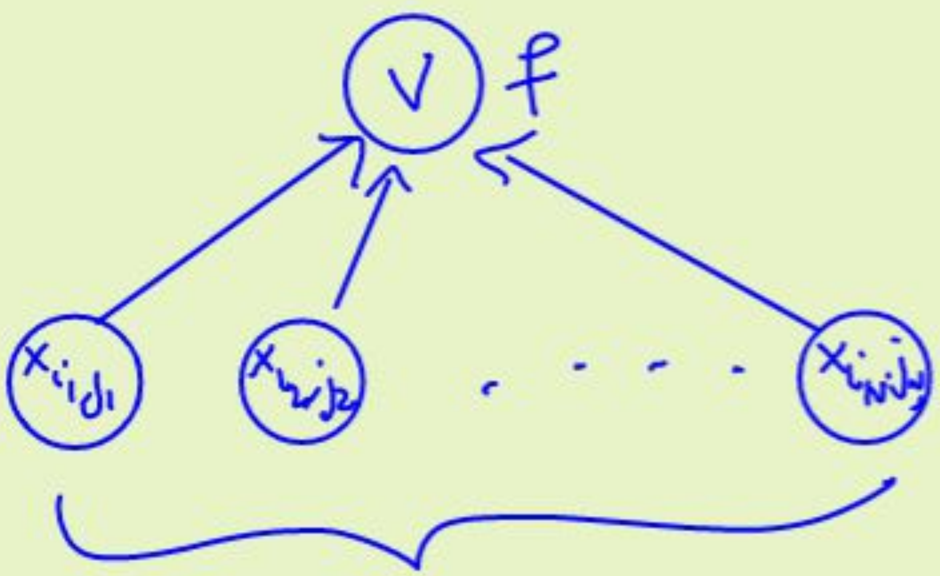

N edges in input graph

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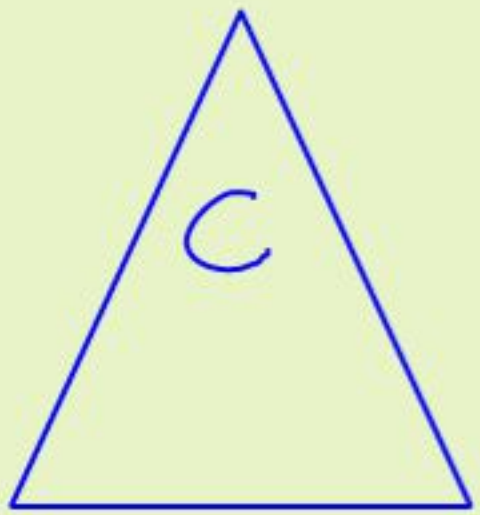

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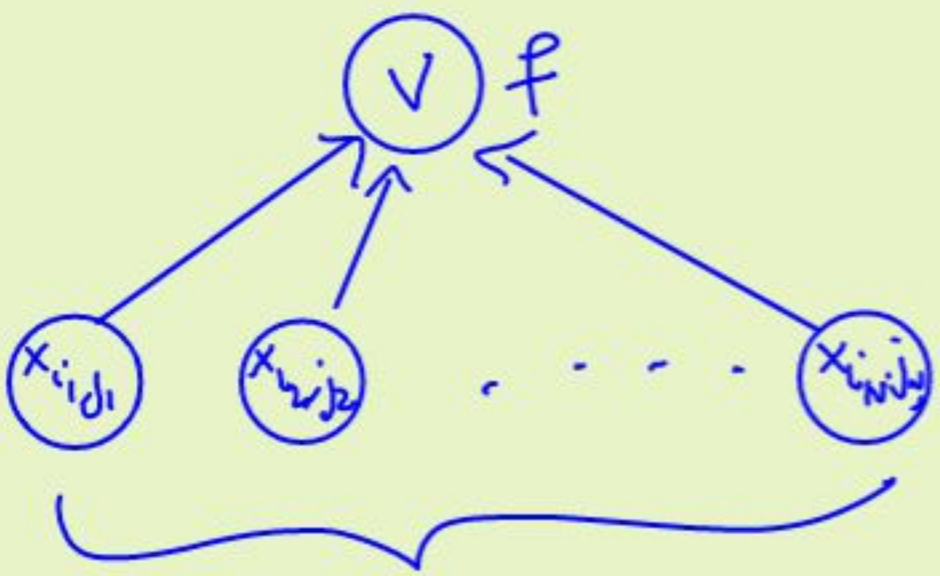

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$\Rightarrow$ Matching/Star of size $\sqrt{N}/2$

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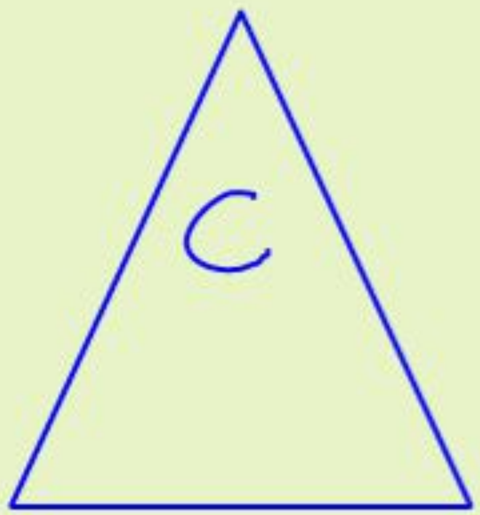

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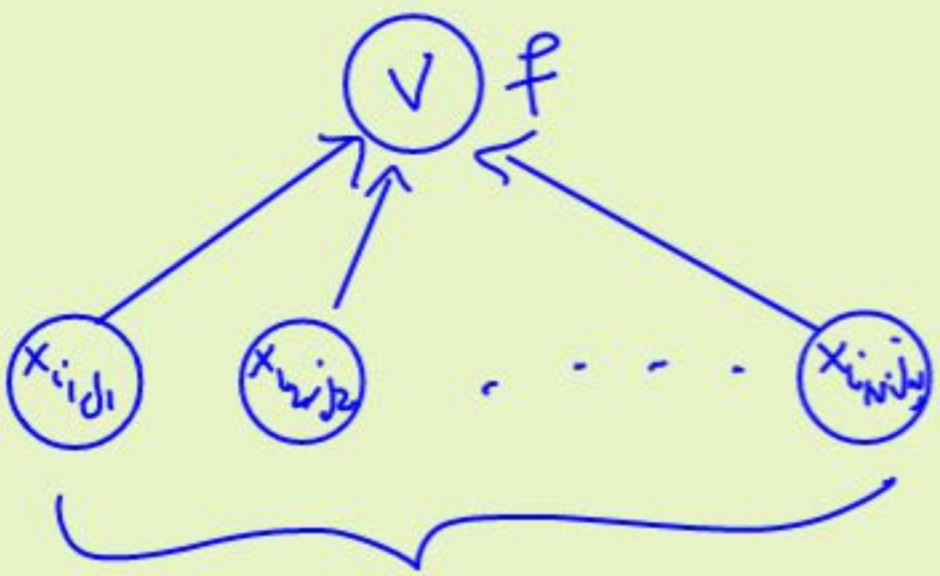

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$$\Pr_{D_0}[f(x)=0] \leq \Pr[\text{all endpts of star/matching on same side of partition}] \leq 2^{-\sqrt{N}/2}$$

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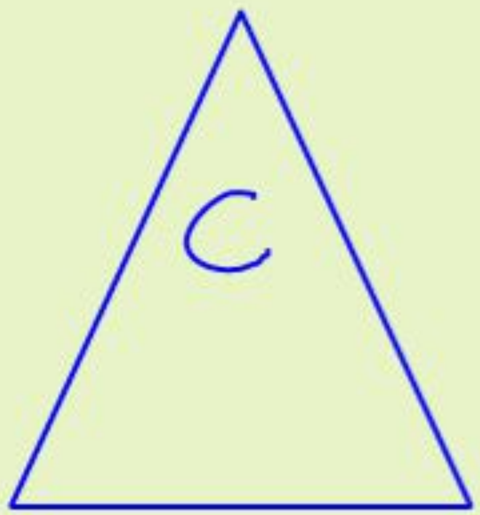

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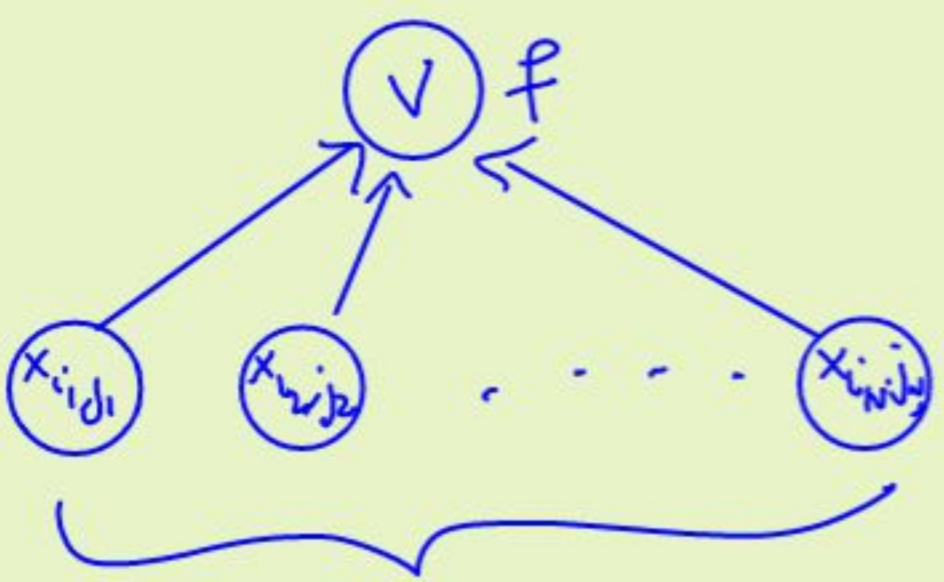

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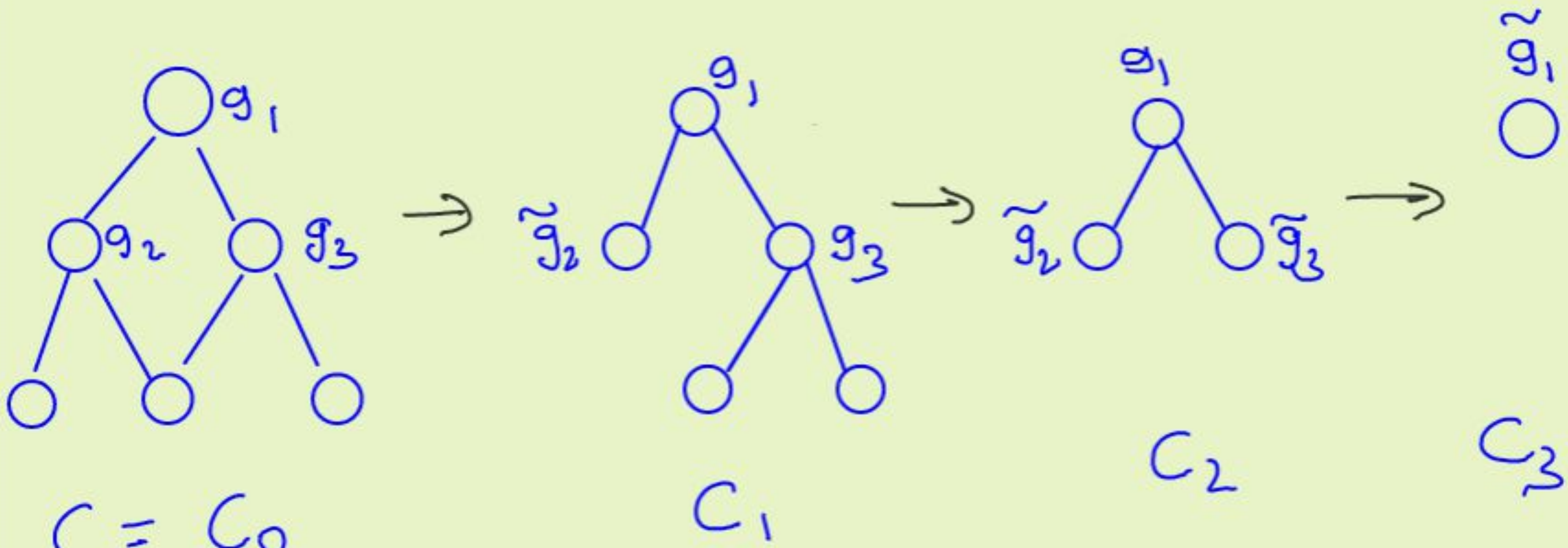
$\Pr_{D_0}[f(x)=0]$ $\leq$ Pr[all endpts of star/matching on same side of partition] $\leq$ $2^{-\sqrt{N}/2}$

$N \geq 100 \log^2 n \Rightarrow$ Can replace f by 1 [no error on D_1 by monotonicity!]

Approximation method



Approximators: $\bigvee_{\{i,j\} \in E} x_{i,j}$

$|E| \le 100 \log^2 n$

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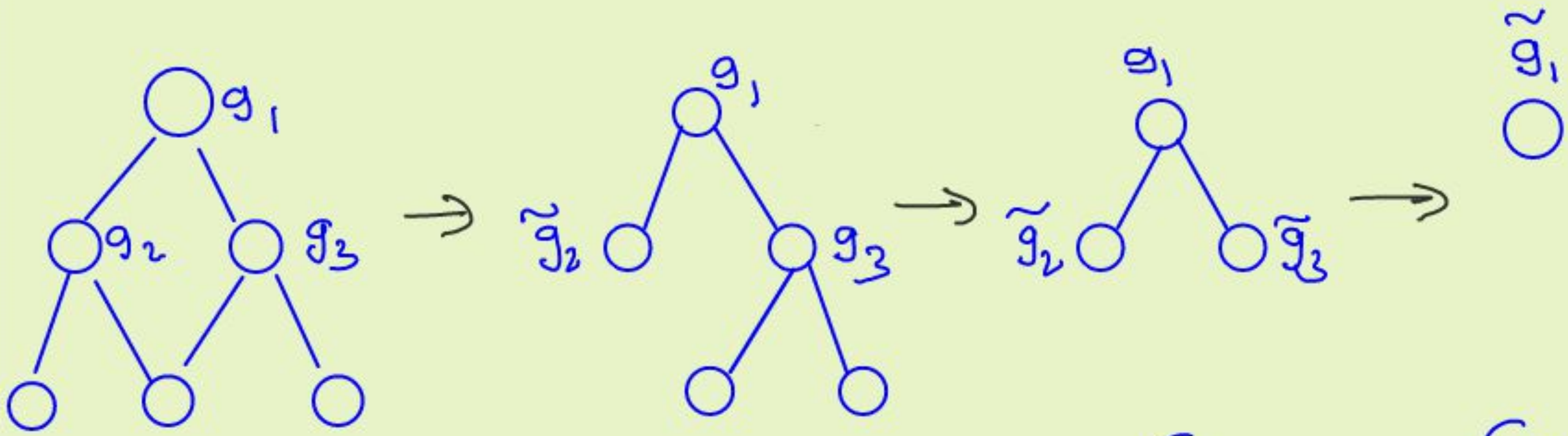

$C = C_0$ $\quad C_1 \quad C_2 \quad C_3$

$$\Pr_{D_0/D_1}\left[C_i(x) \neq C_{i+1}(x) \right] \le \delta < \frac{1}{10s} \implies \Pr_{D_0/D_1}\left[C_0(x) \neq C_s(x) \right] \le s\cdot\delta < 1/10$$

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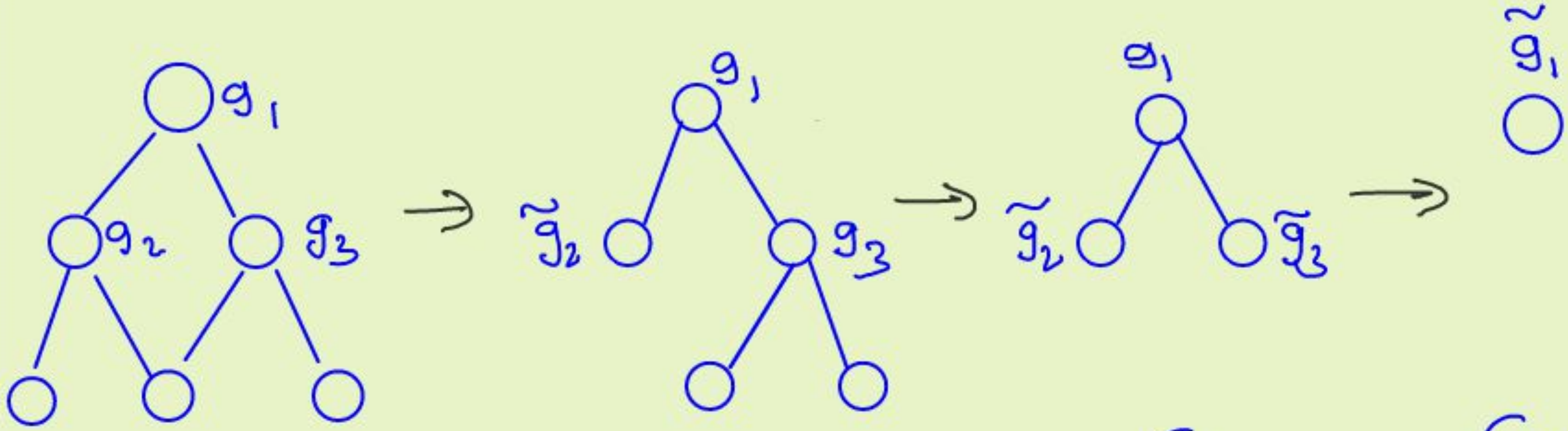

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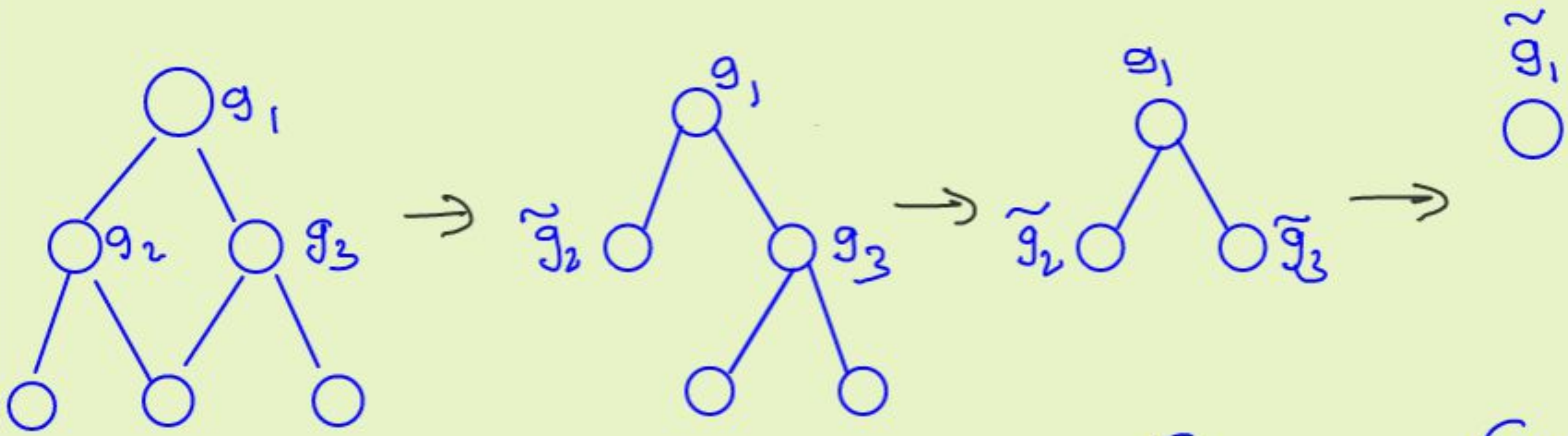

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$|E| \le 100 \log^2 n$ $\qquad = o(1).$

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$$g = \bigvee_{\{i,j\} \in E_1 \cap E_2} x_{i,j} \vee \bigvee_{\{i,j\} \neq \{k,l\}} (x_{i,j} \wedge x_{k,l})$$

$$\tilde{g} = \bigvee_{\{i,j\} \in E_1 \cap E_2} x_{i,j}$$

Obs: $\tilde{g} \le g$ [no errors wrt D_0]

$$\Pr_{D_1}[\tilde{g} \neq g] \le \frac{O(\log^4 n)}{n^3} < \frac{1}{108}$$

$\square$

Larger k, Stronger lower bound

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Sunflower lemma [Erdős-Radó 60]: $\mathcal{F} \subseteq \binom{[n]}{r}$, $|\mathcal{F}| \ge (rp)^r$

$\Longrightarrow$ $\mathcal{F}$ contains sunflower with p petals

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Plucking Sunflowers

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$$\leq \Pr[\bigwedge_{i=1}^{p} h(V_i) \text{ not } 1\text{-}1 \mid h(V_0) \text{ } 1\text{-}1]$$

$\hookrightarrow$ independent!

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$$\le 1/2^p \quad \text{if } |V_i| \le \underbrace{\ell \le \sqrt{k}}.$$

Fix $\ell := \sqrt{k}$.

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$\forall i : \underbrace{|\{V_i \mid |V_i| = r\}| \leq}_{p\text{-small}} (rp)^r \Rightarrow N \leq \underbrace{O((\ell p)^\ell)}_{N_0}$

$$\Pr_{D_1}\left[\bigvee_{i=1}^{N} \mathrm{Cl}_{V_i}(x) = 1\right] \leq \sum_{r=2}^{\ell} \left(\frac{k}{n}\right)^r \cdot (\ell p)^r \leq O\left(\left(\frac{k\ell p}{n}\right)^2\right)$$

$= o(1)$

if $k\ell p < n/2$;

Constructing approximators for each gate

$g = \tilde{g}_1 \vee \tilde{g}_2 \qquad \tilde{g}_1 = \bigvee_{v \in \mathcal{F}_1} \mathrm{Cl}_{a_v}(x), \quad \tilde{g}_2 = \bigvee_{v \in \mathcal{F}_2} \mathrm{Cl}_{a_v}(x).$

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↳ Pluck sunflowers as long as possible.

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$$\Pr_{D_1}[\tilde{g}(x) \neq \hat{\hat{g}}(x)] \leq \sum_{i=\ell+1}^{2\ell} \left(\frac{k}{n}\right)^i (2\ell p)^i \leq O\left(\left(\frac{2\ell k p}{n}\right)^\ell\right)$$

Final lower bound

$$s \geq \Omega\left(\min\left\{\frac{2^p}{O(\ell p)^{2\ell}}, \left(\frac{n}{2k\ell p}\right)^{\ell}\right\}\right)$$

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$$\Rightarrow\ s \geq n^{\Omega(\ell)} = n^{\Omega(\sqrt{k})}.$$

Monotone Real Circuits [Pudlák '97]

→ Monotone Real function $f : \mathbb{R}^a \to \mathbb{R}$ Eg: Max, Min

$(x_1, \ldots, x_a) \leq (y_1, \ldots, y_a) \Rightarrow f(x) \leq f(y)$

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<u>Thm [Pudlák '97]</u>: Same lower bound for $\text{Clique}_{n,k}$ in the monotone real circuit setting.

Changes in the proof

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→ Same hard distributions D_0 & D_1

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$|V_i| \leq \ell$, p-small.

Changes in the proof $\to$ Wlog $C(x) \in [0,1]\ \forall x$

$\to$ Same hard distributions D_0 & D_1

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$|V_i| \leq \ell$, p-small. $\quad \alpha_i \in \mathbb{R}$

$\to$ Plucking sunflowers - $V_1, \ldots, V_p$ s.f. with core V_0

$\alpha_1 \leq \cdots \leq \alpha_p$

Replace $\alpha_1 Clq_{V_1}(x)$ by $\alpha_1 Clq_{V_0}(x)$.

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Changes in the proof $\rightarrow$ Wlog $C(x) \in [0,1]$ $\forall x$

$\rightarrow$ Same hard distributions $\mathcal{D}_0$ & $\mathcal{D}_1$

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$\rightarrow$ No errors wrt. $\mathcal{D}_1$

$\rightarrow$ $\Pr_{\mathcal{D}_0}[\text{error}] \leq \Pr[V_0 \text{ clique} \wedge V_1, \ldots, V_p \text{ not cliques}]$

$\leq 2^{-p}$.

Changes in the proof → Wlog $C(x) \in [0,1]\ \forall x$

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→ l-Approximators - $\bigvee_{i=1}^{N} Cl_{a_{V_i}}(x) \rightsquigarrow \operatorname{Max}_{i=1}^{N} \alpha_i \cdot Cl_{a_{V_i}}(x)$

$|V_i| \leq l$, p-small. $\qquad \alpha_i \in \mathbb{R}$

→ $\varphi\left(\operatorname{Max}_{i=1}^{N} \alpha_i\, Cl_{a_{V_i}}(x), \operatorname{Max}_{j=1}^{M} \beta_j\, Cl_{a_{W_j}}(x) \right)$

φ: monotone real function

Changes in the proof → Wlog $C(x) \in [0,1]\ \forall x$

→ Same hard distributions D_0 & D_1

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$|V_i| \leq l$, p-small.

→ $\underbrace{\varphi}_{\text{monotone real function}} \left(\underset{i=1}{\overset{N}{\mathrm{Max}}}\, \alpha_i \cdot \mathrm{Cla}_{V_i}(x), \underset{j=1}{\overset{M}{\mathrm{Max}}}\, \beta_j\, \mathrm{Cla}_{W_j}(x) \right)$

$= \underset{i,j}{\mathrm{Max}}\ \varphi(\alpha_i, \beta_j)\ \mathrm{Cla}_{V_i}(x) \wedge \mathrm{Cla}_{W_j}(x)$

monotone real function

Later Improvements

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→ Lower bounds via query-to-communication lifting [GGKS'18]