Geometry of tropical varieties

Lecture I

27/06/2022

Motivation

- Study of families of algebraic varieties
- Analytic behavior of complex manifolds under degenerations
- Connection and application to combinatorics

Plan of the course

- Lectures I and II : introduction to tropical geometry
- Lecture III : Geometry of tropical fans
- Lecture IV : Geometry of tropical varieties

§1. Valuations

Definition A valuation on a field K is the data of an application $v: K \to \mathbb{R} \cup \{\infty\}$ which verifies

1. $v(a) = \infty \iff a = 0$
2. $v(ab) = v(a) + v(b)$
3. $v(a+b) \geq \min\{v(a), v(b)\}$.

Examples

- $K = \mathbb{Q}$ $\quad p$ prime number, $\quad v_p$ p-adic valuation
- $K = \mathbb{C}(t) := \left\{ \frac{P(t)}{Q(t)} \mid P, Q \text{ polynomials in } t,\ Q \neq 0 \right\}$

  $$v\left(\frac{P(t)}{Q(t)} \cdot t^k\right) = k \quad \text{given } P(0), Q(0) \neq 0$$
- $K = \mathbb{C}((t)) := \left\{ \sum_{k \geq -n} c_k t^k \mid c_k \in \mathbb{C},\ n \in \mathbb{N} \right\}$ Laurent series
- $K = \bigcup \mathbb{C}((t^{1/n})) = \overline{\mathbb{C}((t))}$ Puiseux series

Definition

Semifield of tropical numbers $\mathbb{T} := (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$

- $a \oplus b := \min\{a, b\}$
- $a \odot b := a + b$.
- ∞ is the neutral element for tropical addition
- 0 is the neutral element for tropical multiplication

Proposition

A valuation on a field K is a morphism of semifields, that is

$\nu : K \to \mathbb{T}$ such that

- $\nu(ab) = \nu(a) \odot \nu(b)$
- $\nu(a+b) \oplus \nu(a) \oplus \nu(b) = \nu(a) \oplus \nu(b)$
- $\nu(0) = \infty$, $\nu(1) = 0$.

Set-up

- k is a base field
- K is a valued field containing k, $K = \overline{K}$
- $k = K$ is allowed

  $v|_k$ is also allowed to be trivial.

typically: $k = \mathbb{C}$

K = field of Puiseux series

$= \bigcup_{n \in \mathbb{N}} \mathbb{C}((t^{1/n}))$

§2. Algebraic varieties with coordinates

Affine case

$\mathbb{A}^n_k$: affine space of dimension n $/k$

Coordinate ring $k[x_1, \dots, x_n]$

$X \hookrightarrow \mathbb{A}^n_k$: Zariski closed subset

Ideal of definition $I \subseteq k[x_1, \dots, x_n]$

Coordinate ring $k[x_1, \dots, x_n]/I$

Very affine case

T^n_k : the open subset of $\mathbb{A}^n_k$

Coordinate ring $k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$

$X \hookrightarrow T^n_k$ Zariski closed subset

Ideal of definition $I \subseteq k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$

Coordinate ring $k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]/I$

For each field $K \supseteq k$, K-point of $\mathbb{A}^n_k$ is a point $x = (x_1, \dots, x_n)$

with $x_1, \dots, x_n \in K$.

$\mathbb{A}^n_k(K) := \{ K\text{-points of } \mathbb{A}^n_k \} = K^n$

$X \hookrightarrow \mathbb{A}^n_k$, K-point of X is a point $x = (x_1, \dots, x_n)$
I ideal of definition

with $x_1, \dots, x_n \in K$ and $f(x) = 0$ for all $f \in I$.

Similarly, K-points of $\mathbb{T}^n_k$ and $X \hookrightarrow \mathbb{T}^n_k$ defined

$\mathbb{T}^n_k(K) = (K^\times)^n$

$X(K) = \{ (x_1, \dots, x_n) = x \mid x_1, \dots, x_n \in K^\times \text{ and } \forall f \in I,\ f(x) = 0 \}$.

§3. Tropicalization

Set-up $\quad k$ base field

K valued field containing k and $K = \overline{K}$

Tropicalization of points

For each K-point $x = (x_1, \dots, x_n)$ of T_K^n,

$$\mathrm{trop}(x) := (\nu(x_1), \dots, \nu(x_n)) \in \mathbb{R}^n$$

Tropicalization of $X \hookrightarrow T_K^n$

$$\mathrm{Trop}(X) := \overline{\{\mathrm{trop}(x) \mid x \in X(K)\}} \subseteq \mathbb{R}^n$$

topological closure of the tropicalizations of points.

Examples

- $n=1$ $\quad X$ is given by a polynomial $f \in k[x_1^{\pm 1}]$.

  $f = x_1^2 - a x_1 + b$

  $X(K)$ has roots of f as elements

  $\mathrm{Trop}(X) = \{ \nu(r), \nu(s) \}$ with r, s roots of f.

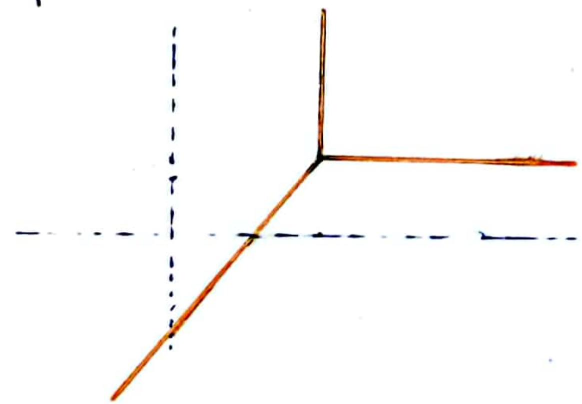
- $n=2$ $\quad X$ given by the equation $X_1 + X_2 = 0$

  $\mathrm{Trop}(X) = \mathbb{R} \subseteq \mathbb{R}^2$, set of points (a,a), $a \in \mathbb{R}$.

- $n=2$ $\quad k = K = \overline{\mathbb{C}((t))}$ $\quad X$ given by equation $X_1 + t X_2 + t^2 = 0$

$x = (x_1, x_2)$, let $\quad a_1 = \nu(x_1)$
$\qquad\qquad\qquad\quad a_2 = \nu(x_2)$

$$\underset{a_1}{X_1} = -\underset{1+a_2}{t X_2} - \underset{2}{t^2}$$



Three cases $\;\Big|\;$ $a_1 = 1 + a_2 \leq 2$
$\qquad\qquad\qquad a_1 = 2 \leq 1 + a_2$
$\qquad\qquad\qquad 1 + a_2 = 2 \leq a_1$

§4. Fundamental theorem

Provides an alternative characterization of tropicalization.

Lemma Let $a, b \in K$, $v(a) \neq v(b) \Rightarrow v(a+b) = \min\{v(a), v(b)\}$.

Proof Assume w.l.g. $v(a) > v(b)$.

- $v(a+b) \geq \min\{v(a), v(b)\} = v(b)$.
- $v(b) = v(a+b-a) \geq \min\{v(a+b), v(a)\} \Rightarrow v(b) \geq v(a+b)$. $\square$

Corollary Assume $a_1 + \cdots + a_m = 0$. Then the minimum of $v(a_1), v(a_2), \ldots, v(a_m)$ is achieved twice.

Corollary Let f be in the ideal I of $X \hookrightarrow T_K^n$.

$f = \sum c_J \underline{x}^J$

$\underline{x}^J = x_1^{j_1} \cdots x_n^{j_n}$

$J = (j_1, \ldots, j_n) \in \mathbb{Z}^n$.

$x \in X(K)$, $\operatorname{trop}(x) = (a_1, \ldots, a_n)$.

$$\Rightarrow \min\{v(c_J) + \underbrace{\langle \operatorname{trop}(x), J \rangle}_{= a_1 j_1 + \cdots + a_n j_n}\} \text{ is achieved twice}$$

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Definition

- For a polynomial $f \in k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, $f = \sum c_{\underline{j}} \underline{x}^{\underline{j}}$,

$$\mathrm{trop}(f) := \bigoplus \nu(c_{\underline{j}}) \odot \underline{x}^{\odot \underline{j}} = \min \{ \nu(c_{\underline{j}}) + \langle \underline{x}, \underline{j} \rangle \}$$

$x_1 j_1 + \dots + x_n j_n$

- "Zero set" of $\mathrm{trop}(f) = \bigoplus \nu(c_{\underline{j}}) \odot \underline{x}^{\odot \underline{j}}$ defined as the set of all $\underline{a} = (a_1, \dots, a_n)$ where the minimum of $\nu(c_{\underline{j}}) + a_1 j_1 + \dots + a_n j_n$ is achieved twice.

Theorem (Fundamental theorem)

$$\mathrm{Trop}(X) = \text{"zero set" of } \mathrm{Trop}(I), \quad \mathrm{Trop}(I) = \{\mathrm{trop}(f) \mid f \in I\}$$

$$= \bigcap_{f \in I} \text{"zero set" of } \mathrm{trop}(f)$$

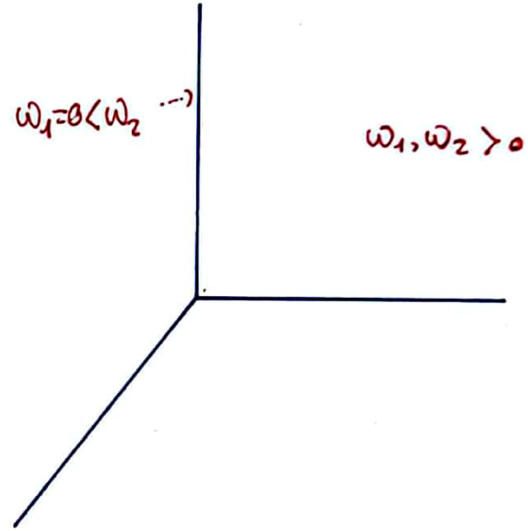
$$= \{ \underline{a} = (a_1, \dots, a_n) \in \mathbb{R}^n \mid \text{for all } f = \sum c_{\underline{j}} \underline{x}^{\underline{j}} \text{ in } I, \text{ the min of } \nu(c_{\underline{j}}) + \langle \underline{a}, \underline{j} \rangle \text{ is achieved twice} \}.$$

Examples

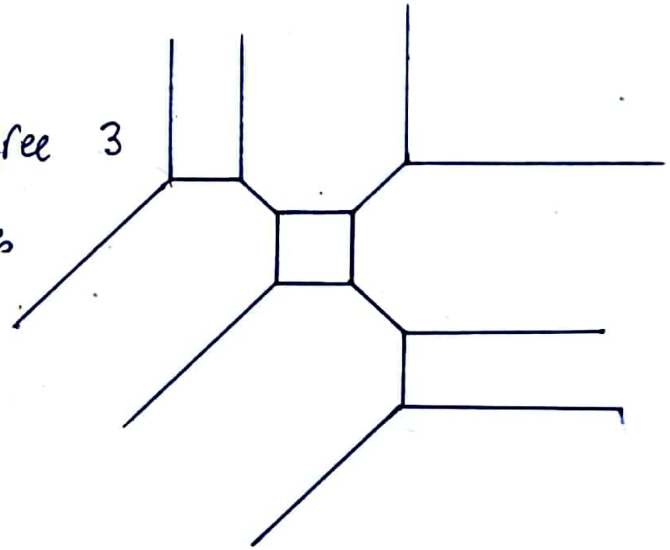
① Let $X \subseteq T_k^2$ given by $f = x_1 + x_2 + 1$

$\text{trop}(f) : \mathbb{R}^2 \to \mathbb{R}$

$(\omega_1, \omega_2) \mapsto \min\{\omega_1, \omega_2, 0\}.$



② Tropicalization of a curve of degree 3 in T_k^2, one of the possibilities



③ Tropicalization of a hypersurface

$X \hookrightarrow T_K^n$ given by $f = \sum c_J \underline{x}^J \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$

Newton Polytope of f $\qquad P := \text{Convex-hull} \langle J \in \mathbb{Z}^n, \ c_J \neq 0 \rangle$.

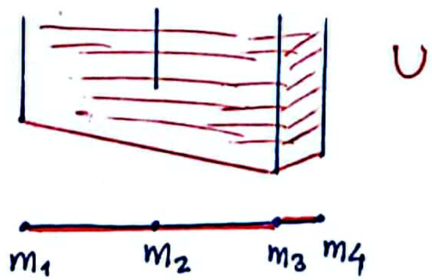
Each c_J gives a height $\nu(c_J)$ to $J \in P$.

This defines a subdivision of P:

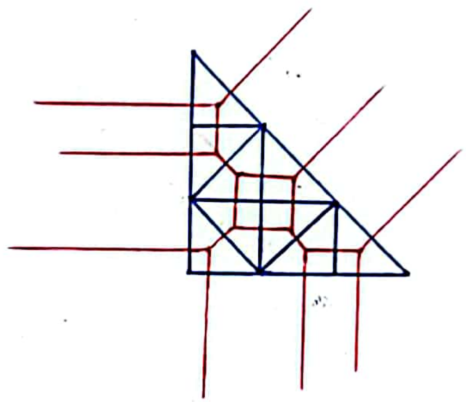
Take $\quad U := \text{Convex-hull} \langle (\alpha, J), \ \alpha \geq \nu(c_J), \ c_J \neq 0 \rangle \subseteq \mathbb{R} \times \mathbb{R}^n$

U is a polyhedron.

bounded faces of U provide the subdivision of P



$\mathrm{Trop}(X)$ is the dual to this subdivision.



④ Hyperplane arrangements

$H_0, H_1, \ldots, H_m \subseteq \mathbb{P}^n_{\mathbb{C}}$ hyperplanes $\quad H_j$ zero set of l_j

$$l_j = \sum_{k=0}^{n} c_k z_k \qquad c_k \in \mathbb{C}$$

$$X := \mathbb{P}^n_{\mathbb{C}} \setminus \bigcup_{j=0}^{n} H_j$$

We get a map $\quad X \to \mathbb{C}^{\times m+1} / \mathbb{C}^{\times} \simeq \mathbb{C}^{\times m}$

$$\underline{x} \in X(\mathbb{C}) \to (l_0(\underline{x}), \ldots, l_m(\underline{x}))$$

This is an embedding $\quad X \hookrightarrow T^{m+1}_{\mathbb{C}} / T^1_{\mathbb{C}} = T^m_{\mathbb{C}}$ if $l_0, \ldots, l_m$ generate the whole space of linear forms.

Equations for X

for $A \subseteq \{0,1,..,m\}$ dependent , that is, such that

$$\sum_{i \in A} c_i \, l_i = 0$$

we get an equation $\sum_{i \in A} c_i X_i = 0$ for $X \hookrightarrow T^{m+1}/T^1$.

Theorem (Ardila-Klivans)

$$\mathrm{Trop}(X) = \bigcap \text{"zero set" of } \bigoplus_{i \in A} X_i \text{ for } A \text{ dependent}$$

$$= \left\{ (a_0, .., a_m) \in \mathbb{R}^{m+1} / \mathbb{R}(1,..,1) \;\middle|\; \min_{j \in A}\{a_j\} \text{ is achieved at least twice} \right\}.$$

Hyperplane arrangement $H_0, \ldots, H_m$ defines a matroid M on the ground set $E = \{0, \ldots, m\}$.

$\mathrm{Trop}(X)$ is the union of cones $\sigma_{\mathcal{F}} \subseteq \mathbb{R}^{m+1} / \mathbb{R}(1, \ldots, 1)$

with $\mathcal{F}: \emptyset \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k \subsetneq E$ flag of flats in M.

$$\sigma_{\mathcal{F}} = \text{convex-cone of } \langle 1_{F_1}, 1_{F_2}, \ldots, 1_{F_k} \rangle$$
$$= \mathbb{R}_+ 1_{F_1} + \cdots + \mathbb{R}_+ 1_{F_k}$$

This gives to $\mathrm{Trop}(X)$ the structure of a rational fan determined by the matroid M.

This is called the Bergman fan of M. (Definition makes sense for any matroid.)