Def a theory is defined by a set of local ops* and all their corr. functions

$$\{ \underbrace{\mathcal{O}_{\Delta, j, \bar{j}, R, r,}}_{\text{in unitary irreps of superconf algebra}} {}_{f} \} , \{ \langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle \}$$

$f \to$ rep. f of global / flavor symm.

global $\to$ commute w/ $SU(2,2|2)$

- unitarity $\Delta > \Delta_B(j,\bar{j},R,r)$ – long irreps

or $\Delta = \Delta_{short}(j,\bar{j},R,r)$ – short / atypical – irreducible components

operator – state correspondance – states in Hilbert space

superconformal algebra acts on Hilbert space.

$$SU(2,2|2) \supset \underbrace{SO(4,2)}_{\text{conformal}} \oplus \underbrace{SU(2)_R \oplus U(1)_R}_{\text{R-symmetry}}$$

$$
\begin{aligned}
[\mathcal{M}_\alpha{}^\beta, \mathcal{M}_\gamma{}^\delta] &= \delta_\gamma{}^\beta \mathcal{M}_\alpha{}^\delta - \delta_\alpha{}^\delta \mathcal{M}_\gamma{}^\beta \,, \\
[\bar{\mathcal{M}}^{\dot\alpha}{}_{\dot\beta}, \bar{\mathcal{M}}^{\dot\gamma}{}_{\dot\delta}] &= -\,\delta^{\dot\alpha}{}_{\dot\delta} \bar{\mathcal{M}}^{\dot\gamma}{}_{\dot\beta} + \delta^{\dot\gamma}{}_{\dot\beta} \bar{\mathcal{M}}^{\dot\alpha}{}_{\dot\delta} \,,
\end{aligned}
$$

Lorentz $M_{\mu\nu} \longrightarrow M_\alpha{}^\beta$ $\quad M_{\dot\alpha}{}^{\dot\beta}$ – adjoint reps of $sl(2,\mathbb{C})$

$so(4,\mathbb{C}) \cong sl(2,\mathbb{C}) \oplus sl(2,\mathbb{C})$

$j \qquad \bar{j}$

$$
\begin{aligned}
[\mathcal{M}_\alpha{}^\beta, P_{\gamma\dot\gamma}] &= \delta_\gamma{}^\beta P_{\alpha\dot\gamma} - \tfrac{1}{2}\delta_\alpha{}^\beta P_{\gamma\dot\gamma} \,, \\
[\bar{\mathcal{M}}^{\dot\alpha}{}_{\dot\beta}, P_{\gamma\dot\gamma}] &= -\,\delta^{\dot\alpha}{}_{\dot\gamma} P_{\gamma\dot\beta} + \tfrac{1}{2}\delta^{\dot\alpha}{}_{\dot\beta} P_{\gamma\dot\gamma} \,, \\
[\mathcal{M}_\alpha{}^\beta, K^{\dot\gamma\gamma}] &= -\,\delta_\alpha{}^\gamma K^{\dot\gamma\beta} + \tfrac{1}{2}\delta_\alpha{}^\beta K^{\dot\gamma\gamma} \,, \\
[\bar{\mathcal{M}}^{\dot\alpha}{}_{\dot\beta}, K^{\dot\gamma\gamma}] &= \delta^{\dot\gamma}{}_{\dot\beta} K^{\dot\alpha\gamma} - \tfrac{1}{2}\delta^{\dot\alpha}{}_{\dot\beta} K^{\dot\gamma\gamma} \,,
\end{aligned}
$$

P, K vectors of Lorentz (1,1)

Dynkin label

$$
\begin{aligned}
[\mathcal{M}_\alpha{}^\beta, Q^i_\gamma] &= \delta_\gamma{}^\beta Q^i_\alpha - \tfrac{1}{2}\delta_\alpha{}^\beta Q^i_\gamma \,, && (1,0) \\
[\bar{\mathcal{M}}^{\dot\alpha}{}_{\dot\beta}, \tilde{Q}_{i\dot\delta}] &= -\,\delta^{\dot\alpha}{}_{\dot\delta} \tilde{Q}_{i\dot\beta} + \tfrac{1}{2}\delta^{\dot\alpha}{}_{\dot\beta} \tilde{Q}_{i\dot\delta} \,, && (0,1)
\end{aligned}
$$

of Lorentz

$$
\begin{aligned}
[\mathcal{M}_\alpha{}^\beta, S_i{}^\gamma] &= -\,\delta_\alpha{}^\gamma S_i{}^\beta + \tfrac{1}{2}\delta_\alpha{}^\beta S_i{}^\gamma \,, \\
[\bar{\mathcal{M}}^{\dot\alpha}{}_{\dot\beta}, \tilde{S}^{i\dot\gamma}] &= \delta^{\dot\gamma}{}_{\dot\beta} \tilde{S}^{i\dot\alpha} - \tfrac{1}{2}\delta^{\dot\alpha}{}_{\dot\beta} \tilde{S}^{i\dot\gamma} \,,
\end{aligned}
$$

$$
\begin{aligned}
[D, P_{\alpha\dot\alpha}] &= P_{\alpha\dot\alpha} \,, && \text{– raises } \Delta \text{ by } 1 \\
[D, K^{\dot\alpha\alpha}] &= -\,K^{\dot\alpha\alpha} \,, && \text{– lowers } \Delta \text{ by } 1
\end{aligned}
$$

$$
\begin{aligned}
[D, Q^i_\alpha] &= \tfrac{1}{2} Q^i_\alpha \,, \\
[D, \tilde{Q}_{i\dot\alpha}] &= \tfrac{1}{2} \tilde{Q}_{i\dot\alpha} \,,
\end{aligned}
$$

raises Δ by $\frac{1}{2}$

$$
\begin{aligned}
[D, S_i{}^\alpha] &= -\,\tfrac{1}{2} S_i{}^\alpha \\
[D, \tilde{S}^{i\dot\alpha}] &= -\,\tfrac{1}{2} \tilde{S}^{i\dot\alpha} \,;
\end{aligned}
$$

lowers Δ by $\frac{1}{2}$

$$
[K^{\dot\alpha\alpha}, P_{\beta\dot\beta}] = 4\delta_\beta{}^\alpha \delta^{\dot\alpha}{}_{\dot\beta} D - 4\delta_\beta{}^\alpha \bar{\mathcal{M}}^{\dot\alpha}{}_{\dot\beta} + 4\delta^{\dot\alpha}{}_{\dot\beta} \mathcal{M}_\beta{}^\alpha
$$

$$
\begin{aligned}
[K^{\dot\alpha\alpha}, Q^i_\beta] &= 2\delta_\beta{}^\alpha \tilde{S}^{i\dot\alpha} \,, \\
[K^{\dot\alpha\alpha}, \tilde{Q}_{i\dot\beta}] &= 2\delta_{\dot\beta}{}^{\dot\alpha} S_i{}^\alpha \,, \\
[P_{\alpha\dot\alpha}, S_i{}^\beta] &= -\,2\delta_\alpha{}^\beta \tilde{Q}_{i\dot\alpha} \,, \\
[P_{\alpha\dot\alpha}, \tilde{S}^{i\dot\beta}] &= -\,2\delta_{\dot\alpha}{}^{\dot\beta} Q^i_\alpha \,,
\end{aligned}
$$

R-symmetry

$SU(2)_R \oplus U(1)_r$:

$\{R^+, R^-, R\}$ $\qquad$ r

$$R^1{}_2 = R^+ \;, \qquad R^2{}_1 = R^- \;, \qquad R^1{}_1 = -\frac{1}{4}r + R \;, \qquad R^2{}_2 = -\frac{1}{4}r - R \;,$$

$$[R^i{}_j, \begin{pmatrix} M, P, K \\ D \end{pmatrix}] = 0$$

$$[R^i{}_j, R^k{}_l] = \delta^k{}_j R^i{}_l - \delta^i{}_l R^k{}_j$$

2 of $SU(2)_R$

$r=-1$
$$[R^i{}_j, Q^k_\alpha] = \delta_j{}^k Q^i_\alpha - \frac{1}{4}\delta^i_j Q^k_\alpha \;,$$

$r=1$
$$[R^i{}_j, \tilde{Q}_{k\dot\alpha}] = -\delta_k{}^i \tilde{Q}_{j\dot\alpha} + \frac{1}{4}\delta^i_j \tilde{Q}_{k\dot\alpha} \;,$$

$$\begin{aligned} [R^i{}_j, \tilde{S}^{\dot\alpha k}] &= \delta_j{}^k \tilde{S}^{\dot\alpha i} - \frac{1}{4}\delta^i_j \tilde{S}^{\dot\alpha k} \;, \\ [R^i{}_j, S^\alpha_k] &= -\delta_k{}^i S^\alpha_j + \frac{1}{4}\delta^i_j S^\alpha_k \;, \end{aligned}$$

$$\begin{aligned} \{Q^i_\alpha, \tilde{Q}_{j\dot\alpha}\} &= \tfrac{1}{2}\delta^i{}_j P_{\alpha\dot\alpha} \;, \\ \{\tilde{S}^{i\dot\alpha}, S_j{}^\alpha\} &= \tfrac{1}{2}\delta^i{}_j K^{\dot\alpha\alpha} \;, \\ \{Q^i_\alpha, S_j{}^\beta\} &= \tfrac{1}{2}\delta^i{}_j \delta_\alpha{}^\beta D + \delta^i{}_j \mathcal{M}_\alpha{}^\beta - \delta_\alpha{}^\beta R^i{}_j \;, \\ \{\tilde{S}^{i\dot\alpha}, \tilde{Q}_{j\dot\beta}\} &= \tfrac{1}{2}\delta^i{}_j \delta^{\dot\alpha}{}_{\dot\beta} D - \delta^i{}_j \bar{\mathcal{M}}^{\dot\alpha}{}_{\dot\beta} + \delta^{\dot\alpha}{}_{\dot\beta} R^i{}_j \;, \end{aligned}$$

- Flavor symmetry – conserved current – adjoint rep of F – flavor symm.

- Most universal op: local SCFT, has a conserved $\Delta = 4$, spin 2 $\underline{\partial_\mu T^{\mu\nu}} = 0$ – conf. primary

— $T^{\mu\nu}$ → current that generates conf. transf.

it is the top component of a short superconf multiplet

superconf primary: $\Delta = 2$ scalar, singlet under $SU(2)_R \oplus U(1)_r$

$Q, \tilde{Q}$

supersymm. currents, $SU(2)_R$, $U(1)_r$ currents

$\downarrow \qquad + Q, \tilde{Q}$

$T_{\mu\nu}$ — top component.

Corr fns are constrained by conf inv.:

$\langle \mathcal{O} \rangle \neq 0$ iff $\Delta_{\mathcal{O}} = 0 \longleftrightarrow \mathcal{O}$ is the identity operator.

$$\langle \mathcal{O}_i(x_1) \mathcal{O}_j(x_2) \rangle = \frac{\delta_{\Delta_i, \Delta_j} C_{ij}}{x_{12}^{2\Delta_i}}$$

$\longrightarrow$ can diagonalize $C_{ij} = \delta_{ij}$ for a unitary theory

$\langle \mathcal{O} \mathcal{O} \mathcal{O} \rangle$ = fixed up to a const.

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = \frac{1}{x_{12}^{2\Delta_\mathcal{O}}\, x_{34}^{2\Delta_\mathcal{O}}}\, g(u,v)$$

identical scalars

conf. inv. cross-ratios

Operator product expansion – finite radius of conv.

$$\mathcal{O}_1(x)\,\mathcal{O}_2(0) \sim \underbrace{\sum_{k \text{ primaries}}} \frac{\lambda_{12k}}{x^{\Delta_1+\Delta_2-\Delta_k}}\left[\mathcal{O}_k(0) + \beta x^\mu \partial_\mu \mathcal{O}_k(0) + \dots\right)$$

fixed by conf sym

+ Spinning conf primaries

→ can be used to reduce n-pt fns to 1-pt fns & comput < - - > from

$\{\lambda_{ijk}\}$ and $\{\mathcal{O}\}$.

$$\big((O_1 O_2) O_3\big) = \big(O_1 (O_2 O_3)\big)$$ – OPA is associative

$\left(\begin{array}{l}\exists \text{ subsector of ops in} \\ \text{4d } \mathcal{N} \geq 2 \text{ SCFTs}\end{array}\right)$ isomorphic to Vertex Operator algebra

4d $\mathcal{N} \geq 2$ SCFTs – certain set of short representations

Vertex Operator algebra – chiral algebra – meromorphic CFT

holomorphic sector of 2d CFT – all ops. depend only on z

$\partial_{\bar{z}} O(z, \bar{z}) = 0$ –

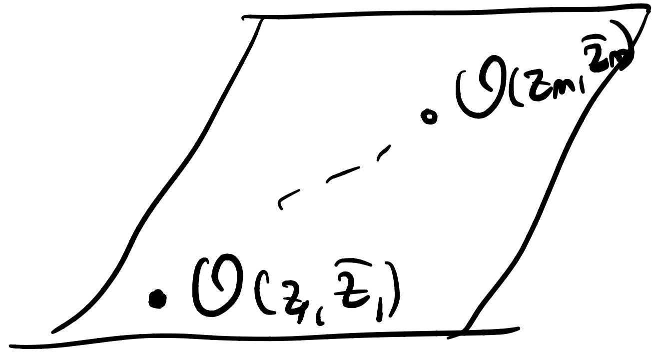
– trivial reps. of the anti holomorphic alg.

VOA of 4d $\mathcal{N}=2$ SCFTs

the claim:

- Pick a plane $\mathbb{R}^2 \subset \mathbb{R}^4$

$x_3 - x_4$ plane



$$\begin{cases} z = x_3 - i x_4 \\ \bar{z} = x_3 + i x_4 \end{cases}$$

- Restrict op. to be on plane

$$\langle \mathcal{O}(z_1, \bar{z}_1) \cdots\cdots \mathcal{O}(z_m, \bar{z}_m) \rangle .$$

- Take "special" ops in certain short reps.

rep of $SU(2)_R$

claim: $u_{I_1}(\bar{z}_1) \cdots u_{I_m}(\bar{z}_m) \langle \mathcal{O}^{I_1}(z_1, \bar{z}_1) \cdots \mathcal{O}^{I_m}(z_m, \bar{z}_m) \rangle$

u_{I_1} ↳ known fns of $\bar{z}$

$= f(\{ z_i \})$

Correlator of a VOA. $\Big\}$ meromorphic fn of z_i

We can fix $f(\{z_i\})$ in terms of finitely many coeffs $\leftrightarrow$ singular pieces of OPE

non-sing. terms of OPE $\longrightarrow$ ∞ many 4d OPE coeffs.

free hyper + free vectors $\longrightarrow$ $T_{\mathfrak{g}}$ $\longleftarrow$ non-Lagrangian theories + free fields

$\uparrow$ Conf man.

Cohomological construction.

VOA plane $(x_3 - x_4)$

$sl(2) \times \overline{sl(2)}$

$$\begin{cases} 2L_{-1} = P_3 + iP_4 \\ 2L_{+1} = K_3 - iK_4 \\ 2L_0 = D - M \end{cases} \qquad \begin{cases} 2\bar{L}_{-1} = P_3 - iP_4 \\ 2\bar{L}_{+1} = K_3 + iK_4 \\ 2\bar{L}_0 = D - M \end{cases}$$

$M = M_+{}^+ - \bar{M}_{\dot{+}}{}^{\dot{+}}$ – rotations on $x_3 - x_4$ plane

$$[\overset{(-)}{L}_m, \overset{(-)}{L}_n] = (m-n)\overset{(-)}{L}_{m+n} \qquad m = 0, \pm 1$$

$\rightarrow$ global conf. alg. in 2d

Some susy preserves the plane:

$$\boxed{Q^i_{-},\ \tilde{Q}_{i\dot{-}},\ S_i^{-},\ \tilde{S}^{i\dot{-}}}$$

$$sl(2) \oplus \overline{sl(2|2)} \subset SU(2,2|2)$$

$$Z = -\frac{r}{2} + M_+ \qquad M_\perp - \text{rots on } x_1 - x_2$$

commutes w/ ↑ – central

Meromophicity – get rid of $\bar{z}$

pass to the cohomology of $\mathbb{Q}$ chosen s.t.

Q and S nilpotent

1. $\mathbb{Q}^2 = 0$ and we want its cohomology, i.e.,

$\mathbb{Q}$-closed $(\mathbb{Q}\mathcal{O} = 0)$ but not $\mathbb{Q}$-exact $(\mathcal{O} \neq \mathbb{Q}X)$

– own subsector

~~2.~~ Keep $\bar{z}$, preserve $sl(2)$ by choice of $sl(2) \oplus \overline{sl(2|2)}$ instead of $sl(2|1) \oplus \overline{sl(2|1)}$

$$[\mathbb{Q}, L_m] = 0 \quad \text{for } m = 0, \pm 1$$

~~3.~~ anti-holomorphic should drop out $- \overline{sl(2)}$

not possible

$$\bar{L}_m = \{\mathbb{Q}, \text{something}\} - \mathbb{Q} \text{ exact}$$

$$\bar{L}_1 = \partial_{\bar{z}} \rightsquigarrow \partial_{\bar{z}} \langle \mathcal{O} \cdots \mathcal{O} \rangle = 0$$

$\exists \ \widehat{sl(2)}$ that is $\mathbb{Q}$-exact instead.

$$\mathbb{Q}_1 = Q^1_- + \underbrace{\zeta}_{\text{phase}} \tilde{S}^{2\dot{-}} \qquad \mathbb{Q}_2 = S_1^- - \underbrace{\frac{1}{\zeta}}_{\text{phase}} \tilde{Q}_{2\dot{-}}$$

$$\underbrace{\mathbb{Q}_1^\dagger}_{\text{radial quant}} = S_1^- + \frac{1}{\zeta} \tilde{Q}_{2\dot{-}} \qquad \mathbb{Q}_2^\dagger = Q^1_- - \zeta \tilde{S}^{2\dot{-}}$$

2 - equivalent choices $\mathbb{Q}_{i=1,2}$ w/ same coho.
and ind of ζ.

there is an $\widehat{sl(2)} = \text{diag}(\overline{sl(2)}, su(2)_R)$ is $\mathbb{Q}$-exact

$$-\{\mathbb{Q}_1, \tilde{Q}_{1\dot{-}}\} = \zeta\{\mathbb{Q}_2, Q^2_{-}\} = \hat{L}_{-1}\,, \quad = \bar{L}_{-1} - \zeta R_-$$

$$-\frac{1}{\zeta}\{\mathbb{Q}_1, S_2^{-}\} = -\{\mathbb{Q}_2, \tilde{S}^{1\dot{-}}\} = \hat{L}_1\,, \quad = \bar{L}_{+1} + \frac{1}{\zeta} R_+$$

$$\{\mathbb{Q}_1, \mathbb{Q}_1^\dagger\} = \{\mathbb{Q}_2, \mathbb{Q}_2^\dagger\} = \hat{L}_0\,, \quad = \bar{L}_0 - R$$

$\bar{L}_n$ are $\overline{sl(2)}$

$\{R, R_+, R_-\}$
$su(2)_R$.

$U_I(\bar{z})$ — implementing twisted transl.
$su(2)_R$

Is the $\mathbb{Q}_i$-coho non-trivial.

What is coho at the origin $\mathcal{O}(0) \longleftrightarrow |0\rangle$

$$\underline{Z} = \{\mathbb{Q}_1, \mathbb{Q}_2\}$$

both Z, $\hat{L}_0$ commute w/ each other and w/ $\mathbb{Q}_i$ — look at Z, $\hat{L}_0$ eigenspaces

also $\mathbb{Q}$-exact — need zero eigenspace.

Operators in coho obey:

$$\Delta - \frac{1}{2}(\underbrace{j + \bar{j}}_{\text{Dynkin labels}}) - R_D = 0, \quad -r + (j - \bar{j}) = 0$$

— harmonic rep. of $\mathbb{Q}_i$-coho at origin. — Schur ops.)

For VOA this is the space of states $\mathcal{V}$

$\mathcal{O}(0) \leftrightarrow |\mathcal{O}\rangle$

Translate from origin s.t. remain in coho:

$$\mathcal{O}(z, \bar{z}) = e^{zL_{-1} + \bar{z}\hat{L}_{-1}} \mathcal{O}(0,0) e^{-zL_{-1} - \bar{z}\hat{L}_{-1}}$$

$$\mathcal{O}(z) = [\mathcal{O}(z, \bar{z})]_{\mathbb{Q}}$$ cohomology classes of $\mathbb{Q}$ are the vertex ops.

$Y(|\mathcal{O}\rangle, z)$ → Meromorphic OPE algebra

$\hat{\mathcal{V}}$ → Vertex operator alg. ← Virasoro element is there.