The finite part of ∞

$$1 + 2 + 3 + 4 + \ldots$$

$$= -\frac{1}{12}$$

Joseph Samuel (ICTS,RRI)

Vigyan Adda Oct 24 , 2021

$$1 + 2 + 3 + 4 + \ldots = -\frac{1}{12}$$

Have you seen this equation before?
Answer yes or no

$$1 + 2 + 3 + 4 + \ldots = -\frac{1}{12}$$

Seems wrong on so many levels

Adding integers to get a fraction

Adding positive numbers to get a negative number

SCAM?

NO!

- Deep mathematics of divergent series
- Techniques useful in theoretical physics
- Can make predictions about real experiments in the laboratory

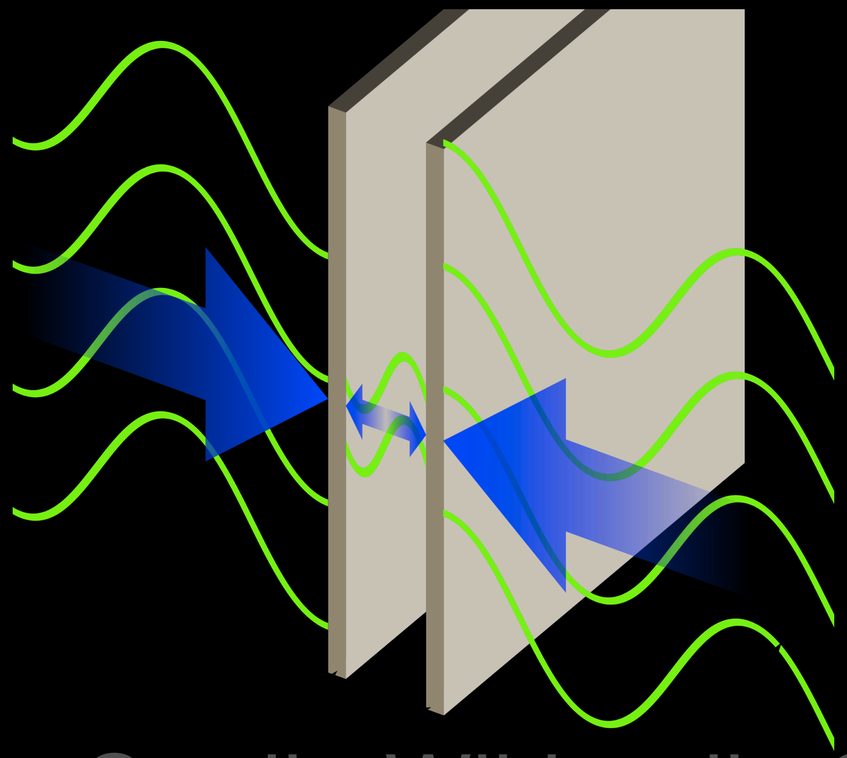

Image Credit: Wikipedia Commons

- The Casimir effect: neutral conducting plates attract each other

Simple example of how we make sense of infinities that appear in physical theories

Strings d=26

The finite part of infinity

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Convergent and Divergent Series

A.P (Arithmetic Progression): 1, 2, 3, 4,……

G.P (Geometric Progression): 1,1/ 2, 1/4, 1/8,……

A.S (Arithmetic Series): 1+2+ 3+ 4+……

G.S (Geometric Series): 1+ 1/2+ 1/4+ 1/8+……

Progression → Sequence

$$\{a_1, a_2, a_3 \ldots\ldots\} = \{a_n \,|\, n = 1,2,3....\infty\}$$

Partial Sums:

$$S_N = \sum_{i=1}^{N} a_i$$

Arithmetic series:

$$S_N = \frac{N(N+1)}{2}$$

Geometric Series:

$$S_N = \frac{1 - r^{(N+1)}}{1 - r} = 1 + r + r^2 + r^3 \ldots r^N$$

Finite Series well defined

Can we assign a value to infinite series?

Zeno

Mathematicians like Euler assumed it was possible

Ramanujam played with infinite series

Cauchy: yes, if the series ``converges”

i.e. the sequence $\{S_N\}$ of partial sums tends to a finite limit

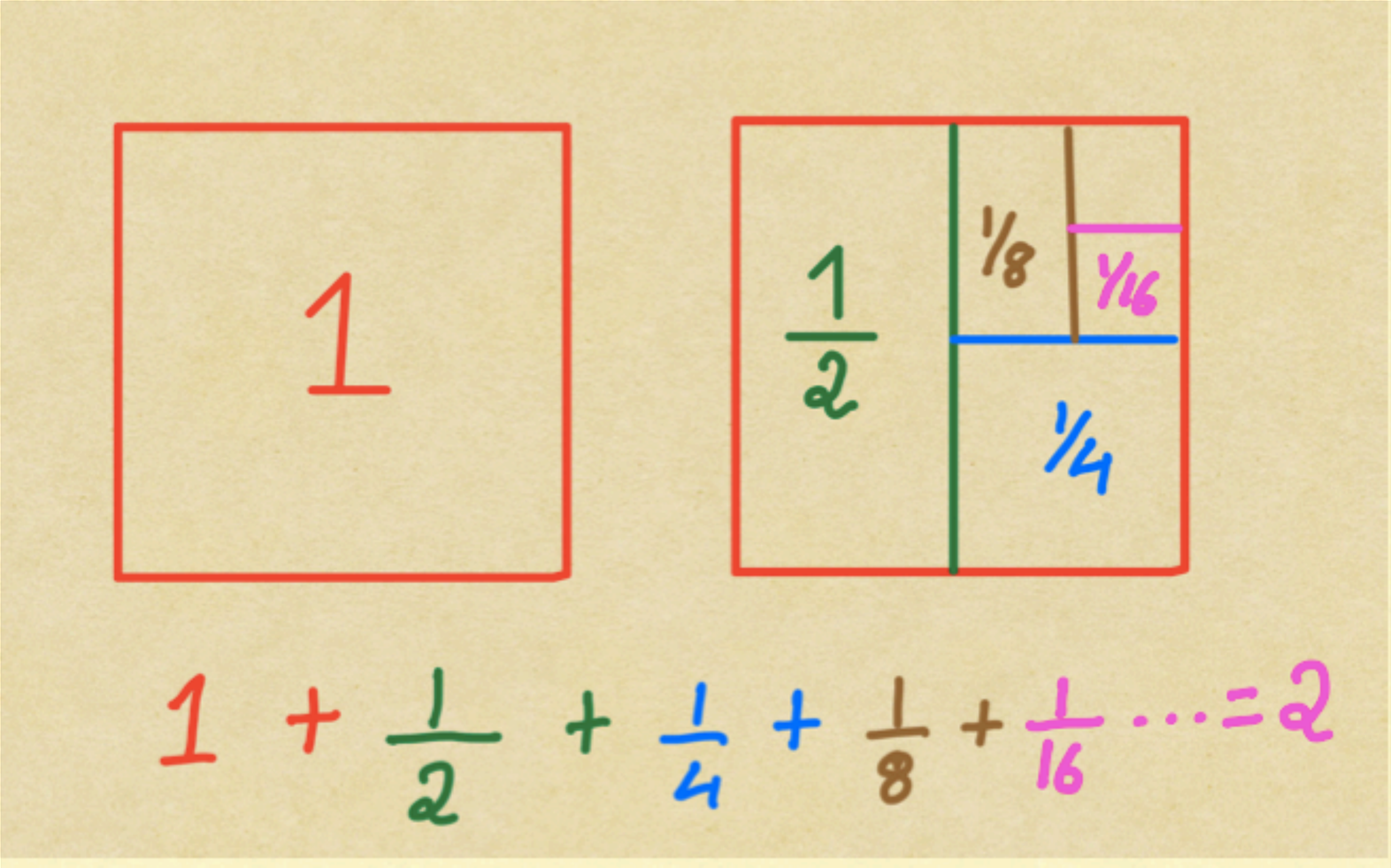
Arithmetic series does not converge

Geometric series converges if $|r| < 1$

Zeno

$$S^G = \lim_{N\to\infty} \frac{1 - r^{(N+1)}}{1 - r}$$

$$= \frac{1}{1 - r}$$



Ramanujam: pushing boundaries. Let r approach -1

$$S^{Gr} = \frac{1}{1-(-1)} = \frac{1}{2} = 1 - 1 + 1 - 1 + 1...$$

More formally,

$$S_N = \sum_{i=1}^{\infty} (-1)^n \rightarrow \sum_{i=1}^{\infty} (-1)^n (\exp - \lambda n)$$

Let λ tend to zero from positive values

Cesaro

$$S = 1 + r + r^2 + \ldots = \frac{1}{1-r}$$

$\mathbb{C}$

$$S.S = (1 + r + r^2 + \ldots)(1 + r + r^2 + \ldots)$$

$$\left(\frac{1}{1-r}\right)^2 = 1 + 2r + 3r^2 + 4r^3 + \ldots$$

$$r \to -1$$

$$\frac{1}{4} = 1 - 2 + 3 - 4 + 5 + \ldots\ldots\ldots$$

$$S^R = 1 + 2 + 3 + 4 + 5 + 6 + \ldots.$$

$$4\,S^R = \quad + 4 \qquad + 8 \qquad + 12$$

$$-3\,S^R = 1 - 2 + 3 - 4 + 5 - 6 \ldots\ldots = \frac{1}{4}$$

$$\boxed{1 + 2 + 3 + 4 + \ldots = -\frac{1}{12}}$$

Alternatively directly force convergence

$$\sum_{n=1}^{\infty} n \rightarrow \sum_{n=1}^{\infty} n \exp - n\lambda$$

$$f(\lambda) = -\frac{d}{d\lambda} \sum_{n=1}^{\infty} \exp - n\lambda$$

$$= \frac{1}{\exp \lambda - 1} = \frac{1}{\lambda^2} - \frac{1}{12} + \frac{\lambda^2}{240} + \ldots\ldots$$

Alternatively directly force convergence

$$\sum_{n=1}^{\infty} n \to \sum_{n=1}^{\infty} n \exp - n\lambda$$

$$f(\lambda) = -\frac{d}{d\lambda} \sum_{n=1}^{\infty} \exp - n\lambda$$

$$= \frac{1}{\exp \lambda - 1} = \frac{1}{\lambda^2} - \frac{1}{12} + \frac{\lambda^2}{240} + \ldots$$

Cautionary Note:
Manipulating divergent series without a valid licence can be injurious to your mental health!

$$(1-1)+(1-1)+\ldots=0$$

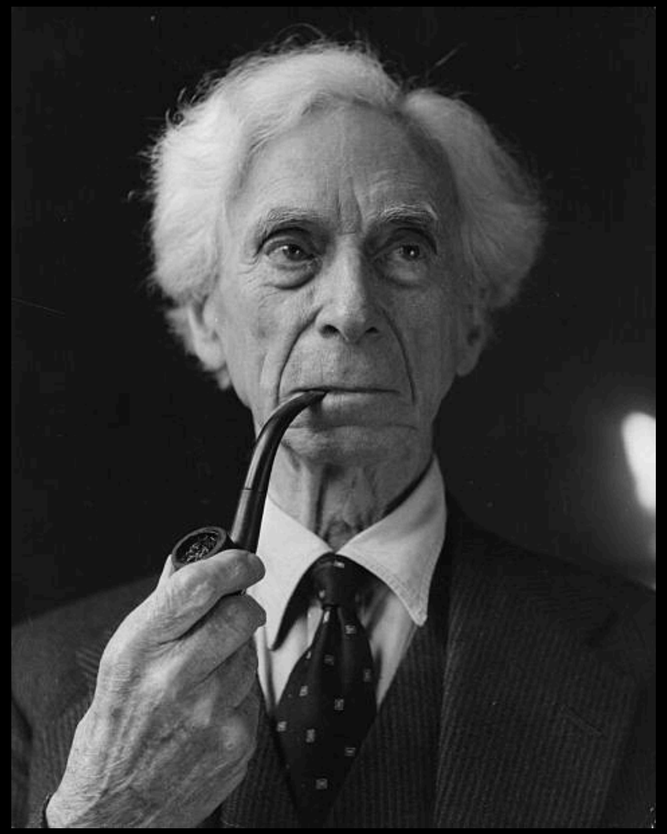
$$1+(-1+1)+(-1+1)+\ldots=1$$

CONTRADICTION!

Walt Whitman: Do I contradict myself?
Very well, then, I contradict myself;
(I am large I contain multitudes).

Bertrand Russell:
Given a contradiction, one can prove anything!

Friend: All right Bertrand
I'll give you that 1=0, prove to me that you are the pope



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Friend: All right Bertrand
I'll give you that 1=0, prove to me that you are the pope

Russell: right, start with 1=0,
add 1 to both sides,
get 2=1

Bertrand Russell:
Given that 1=0, one can prove anything!

1=0, add 1 to both sides get 2=1

I am one and the Pope is one
Together we are two

Bertrand Russell:
Given that 1=0, one can prove anything!

1=0, add 1 to both sides get 2=1
I am one and the Pope is one
Together we are two

But two equals one
So the Pope and I are one! Q.E.D

To avoid contradictions, one needs a consistent theory of divergent series based on general axioms One such is given by complex analysis and analytic continuation $\mathbb{C}$

For instance the Riemann zeta function is defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

For $\Re\ s > 1$. The function diverges for s=1. Extension to the rest of the complex plane by analytic continuation

?

Casimir Effect

Some Quantum Physics

Simple Harmonic oscillator

Classically has lowest energy at rest at the bottom of the harmonic potential

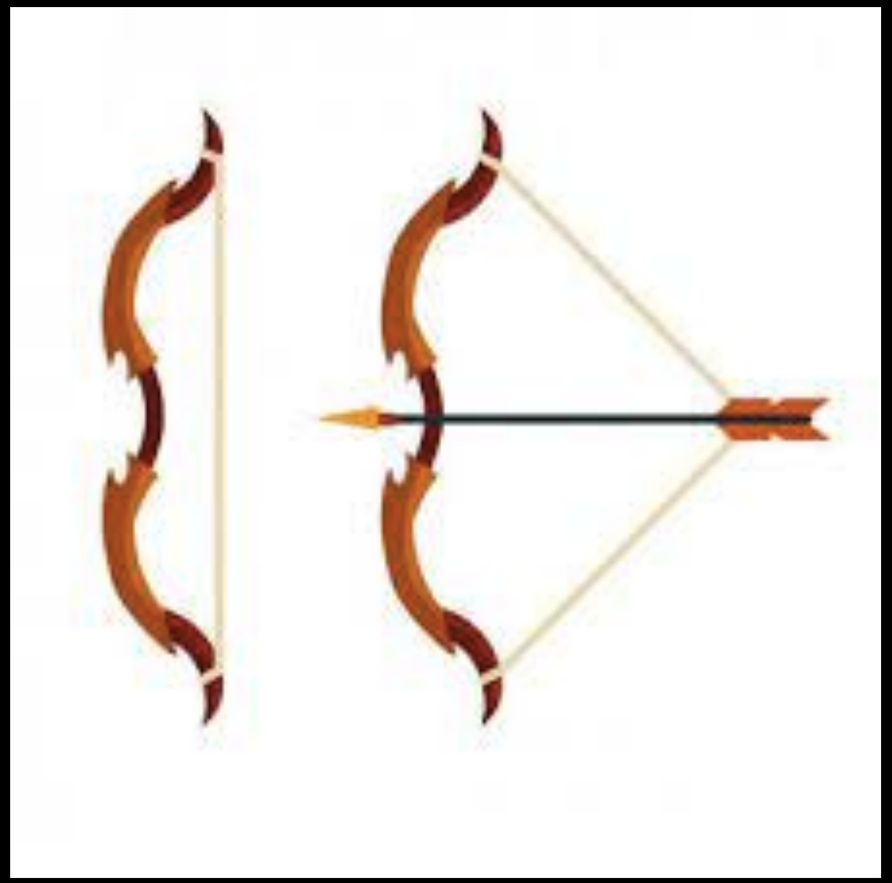
Quantum mechanically forbidden by the uncertainty principle; zero point energy of $\frac{\hbar\omega}{2}$

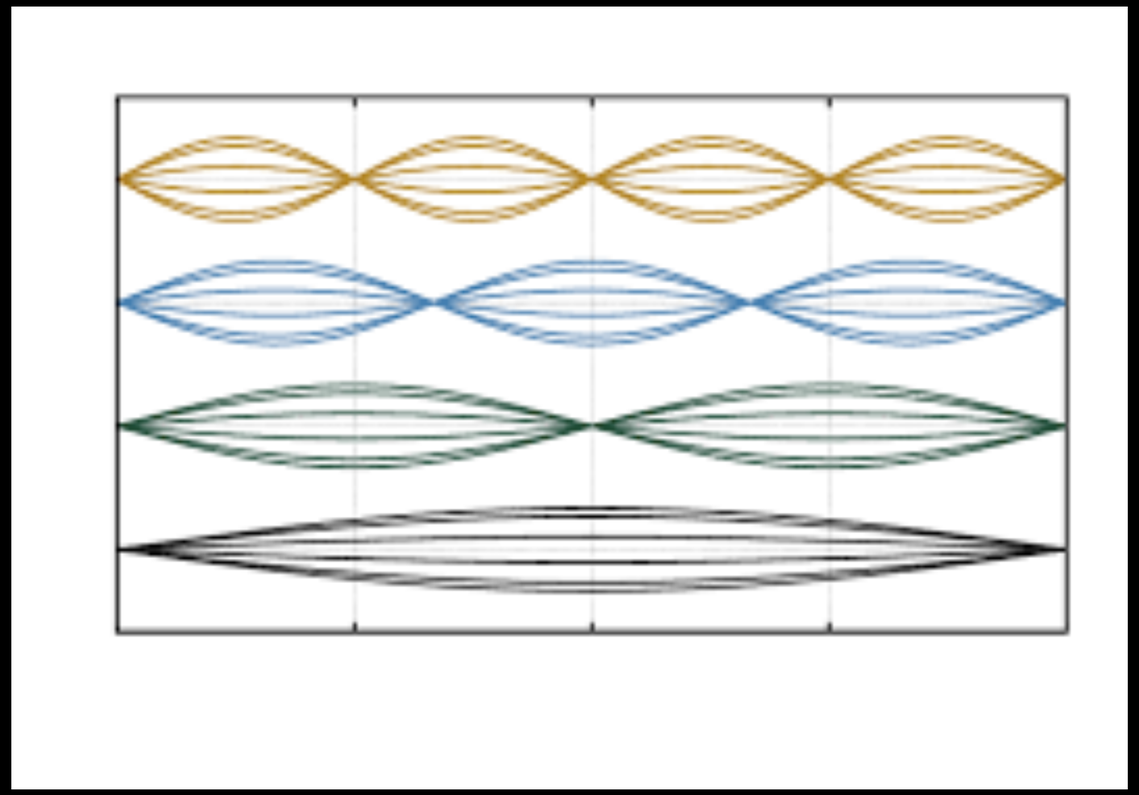
check dimensions!

Casimir Effect

Bow String is Taut

A Plucked string pulls the ends together





Each mode is an oscillator
There are an infinity of modes!

thermal and quantum fluctuations

Casimir Effect

Fundamental frequency $\omega_1 = c_s \frac{\pi}{L}$

Speed of sound is $c_s = \sqrt{\frac{T}{\rho}}$ the square root of the

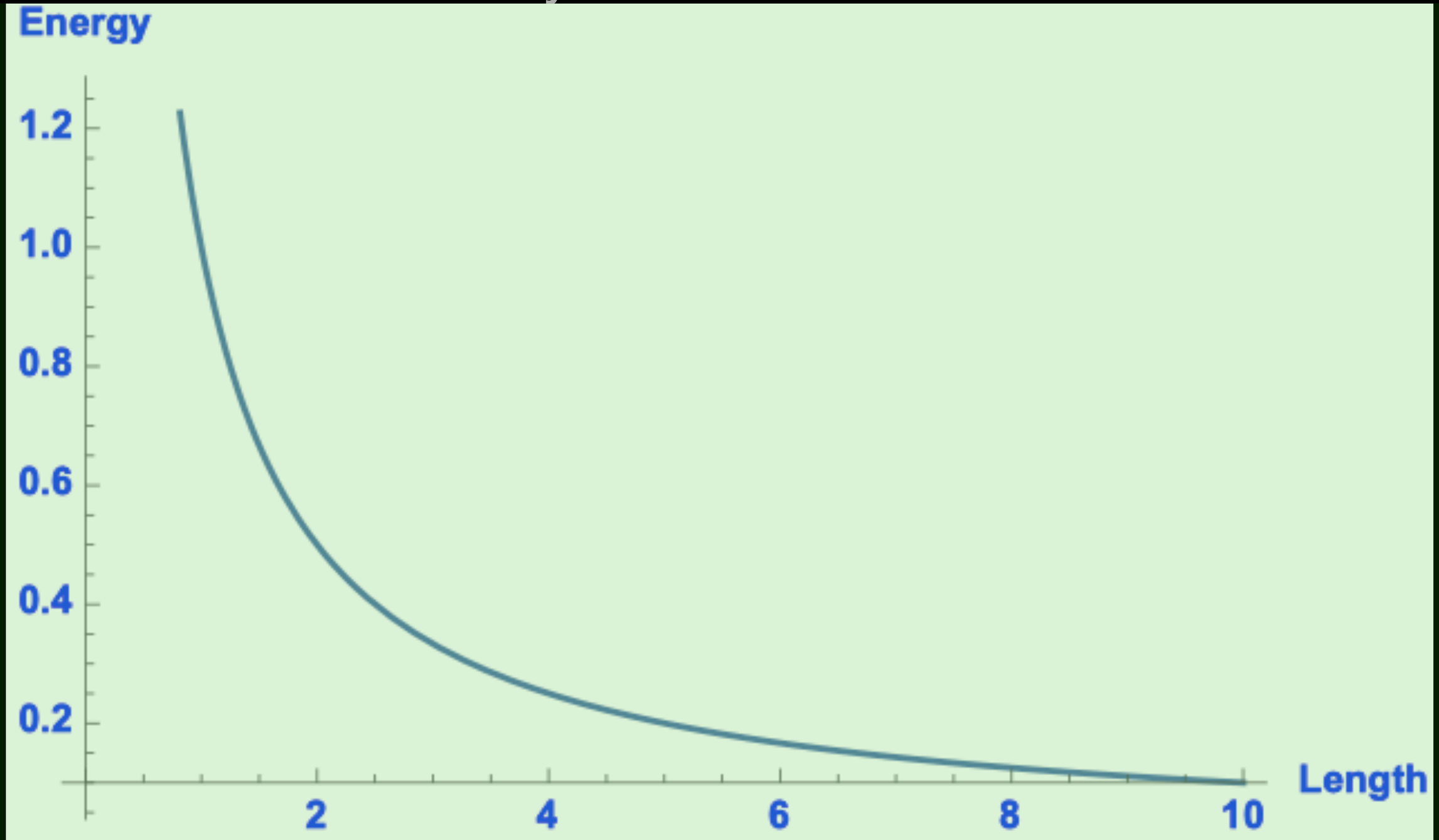
string tension divided by the mass per unit length

Harmonics $\omega_n = n\omega_1 = nc_s \frac{\pi}{L} = nc_s k_1$

2?

Energy in zero point motion is $E(L) = \hbar c_s \frac{\pi}{L} \sum_{n=1}^{\infty} n$

Casimir Effect Theory



Casimir Effect Theory

Seems to diverge AND have the wrong sign

Infinite number of oscillators one for each n

A real string consists of a finite number of atoms!

Sum needs a REGULATOR

$$E(L) = \hbar c_s \pi \sum_{n=1}^{\infty} \frac{n}{L} \rightarrow E(L,\lambda) = \hbar c_s \pi \sum_{n=1}^{\infty} \frac{n}{L} \exp[-\lambda \frac{n}{L}]$$

Gives $\frac{\hbar c_s \pi L}{\lambda^2} - \frac{1}{12}\frac{\hbar c_s \pi}{L} + \frac{\lambda^2}{240L} + \ldots$

We only measure forces in this experiment

Only differences of energy

what happens when L goes to infinity?

Define $x = \frac{n}{L}$ $\qquad dx = \frac{1}{L}$

$$E(L \to \infty, \lambda) = \hbar c_s \pi L \int_0^\infty dx\, x \exp - \lambda x$$

$$= \boxed{\frac{\hbar c_s \pi L}{\lambda^2}}$$

exactly cancels the divergent term in the sum!

Casimir Effect Theory

$$E(L,\lambda) = \boxed{\frac{\hbar c_s \pi L}{\lambda^2}} - \frac{\hbar c_s \pi}{12L} + \frac{\lambda^2}{240L} + \dots$$

$$E(L \to \infty, \lambda) = \boxed{\frac{\hbar c_s \pi L}{\lambda^2}}$$

Difference finite and of negative sign

$$F = \hbar c_s \frac{\pi}{L^2} \zeta(-1)$$

Attractive force

Mechanical system to Electromagnetic system
Argument is virtually identical

Metal wave guide of length L

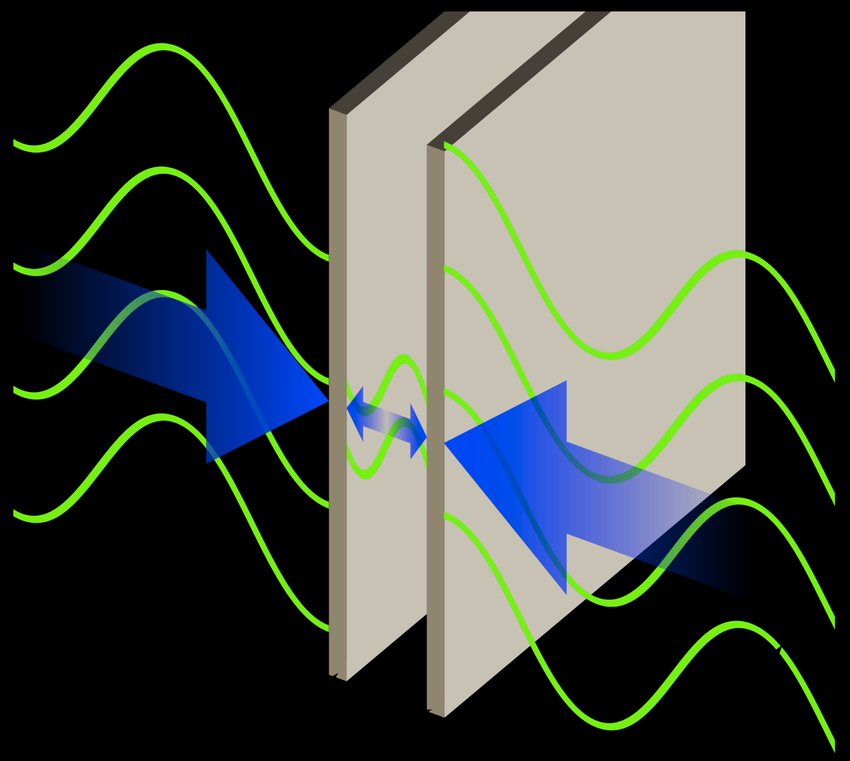
speed of sound replaced by speed of light c

very high frequencies pass through the tube
only low frequency modes should be counted

$$\zeta(-1) = \sum_{n=1}^{\infty} n = -\frac{1}{12}$$

$$F = \hbar c \, \frac{\pi}{L^2} \, \zeta(-1)$$

Casimir Effect Theory



Parallel plates of Area A
and separation L

very similar calculation except

that we have $\sum_{n=1}^{\infty} n^3 = \zeta(-3) = \frac{1}{120}$

$$E(L) - E^{\infty} = -\frac{\hbar c A \pi^2}{6L^3}\zeta(-3)$$

$$F = \frac{\hbar c A \pi^2}{240 L^4}$$

attractive

Exact form of the regulator is not important

Euler-McLaurin Series gives the difference
between the sum and the integral
Integration by parts Bernoulli numbers

Casimir Effect Experiment

HBG Casimir 1948 Theory for Electromagnetic systems

M Sparnay 1958 Parallel plates large uncertainties

SK Lamoreaux 1997 Plate and sphere 5%

U Mohideen and A Roy 1997 Plate and sphere 1%
Atomic Force microscope

Bressi et al 2001 Parallel plates microresonators

IC chip technology electron beam
Lithography no need for any extra alignment
2013-2021 HK, UF, Harvard….

conclusion

We started with the mathematics of divergent series

Physics of the Casimir effect

chose a simple mechanics example

same ideas work for the electromagnetic field

quantum fluctuations have energy

Thermal fluctuations also do

$$\sum_{n=1}^{\infty} k_B T$$

Derivative of energy gives the Casimir Force

Computed the difference between
a sum and an integral
Let’s get a closer look at the integral

Start with a simple example: the thermal fluctuations
of a membrane
Just like a string, a membrane can also vibrate:
drum, lipid membranes, cell walls

A drum has different modes of vibration just like a string

Integral over modes of $k_B T$

$$E(T, A) = \int \frac{d^2x \; d^2k}{(2\pi)^2} \; k_B T \propto A \; k_B T \int_0^\lambda k dk$$

Energy proportional to area;
Energy cost for creating more area
Surface Tension

$$\sigma := \frac{E(T, A)}{A} = \frac{k_B T}{\lambda^2}$$

$$k_B T = 1/40 \text{ ev} \qquad \lambda = 3 \times 10^{-10} \text{metres}$$

$$\sigma \approx 40 \, \frac{\text{milli joules}}{\text{metre}^2}$$

roughly the right order of magnitude for
oil water interface

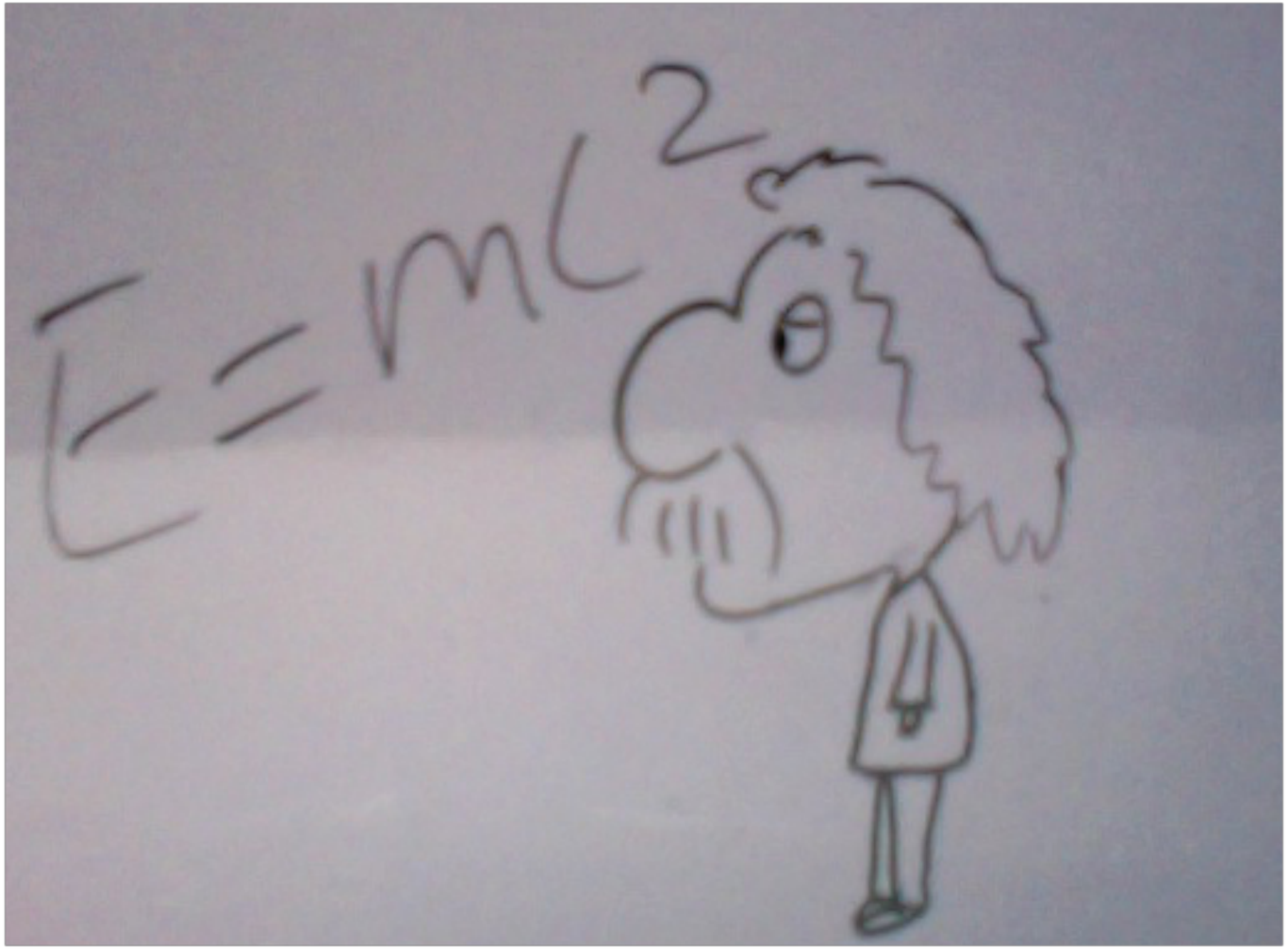
Conclusion

To extract a finite answer for the Casimir force, we invoked two main ideas

The first was the atomistic nature of matter: the string, the waveguide, the plates

Second that only differences of energy are measured in the experiment: Casimir force not energy!

True of THIS experiment; Not all



Energy has Mass

cartoon by ROSHNI SAMUEL

Mass Gravitates!

The integral we subtracted can have gravitational effects

It was formally infinite, but we know that because of the atomistic nature of matter, it is really finite (as all physically measurable quantities should be)

Conclusion

Vacuum Energy of space is repulsive order of magnitude

Repulsive Energy density $\frac{\hbar c}{\lambda^4}$

Atomistic nature of spacetime at a fundamental level

Planck Length $\lambda = 10^{-35}\text{metres}$

Theory predicts a cosmological constant, repulsive force that accelerates the expansion of the Universe

Experimentally seen but value is 10^{-120} times smaller

WHY?

WIDE OPEN PROBLEM FOR THE 21 st Century

THANK YOU

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