

(3) (9)

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For mixed states : $\rho^2 \neq \rho$ ρ is not a projection operator.

Hence $\text{Tr} \rho^2 \neq \text{Tr}(\rho) = 1$

For mixed states $\text{Tr} \rho^2 < 1$.

Thus for a general state pure/mixed we have $\boxed{\text{Tr} \rho^2 \leq 1}$. Can be used as a criterion to find out if a state is a pure or a mixed state.

pure: $\text{Tr} \rho^2 = 1$

mixed: $\text{Tr} \rho^2 < 1$

From def.

$$\rho = \sum_k p_k \rho_k$$

for any ket $|u\rangle$

$$\langle u | \rho | u \rangle = \sum_k p_k \langle u | \rho_k | u \rangle$$

Density operator is a +ve operator \Leftrightarrow
$$\begin{aligned} &= \sum_k p_k \langle u | \psi_k \rangle \langle \psi_k | u \rangle \\ &= \sum p_k |\langle u | \psi_k \rangle|^2 \geq 0 \end{aligned}$$

Physical meaning of matrix elements of ρ in basis $|u_n\rangle$

Diagonal elements :
$$\begin{aligned} \rho_{nn} &= \langle u_n | \rho | u_n \rangle \\ &= \sum_{k \neq n} p_k \langle u_n | \psi_k \rangle \langle \psi_k | u_n \rangle \\ &= \sum_{k \neq n} p_k |\langle u_n | \psi_k \rangle|^2 \end{aligned}$$

Express $|\psi_k\rangle$ in basis $|u_n\rangle$ as

$$|\psi_k\rangle = \sum_n c_n^{(k)} |u_n\rangle \quad c_n^{(k)} = \langle u_n | \psi_k \rangle$$

$$\rho_{nn} = \sum_k p_k |c_n^{(k)}|^2 \Rightarrow \text{positive real numbers}$$

$|c_n^{(k)}|^2$ — probability of $|u_n\rangle$ in pure state $|\psi_k\rangle$
 $\Rightarrow f_{nn}$ — probability of $|u_n\rangle$ in state f
 \Rightarrow diagonal matrix elements are called population of the state $|u_n\rangle$

Physically if N times the same experiment is carried out with the same initial conditions, (N is large)
 Then $\Rightarrow N f_{nn}$ systems will be found in the state $|u_n\rangle$.

* Consider now the off-diagonal elements

$$f_{nm} = \sum_k p_k \langle u_n | f | u_m \rangle$$

$$= \sum_k p_k \langle u_n | \psi_k \rangle \langle \psi_k | u_m \rangle$$

$$= \sum_k p_k c_n^{(k)} c_m^{(k)*}$$

$c_n^{(k)} c_m^{(k)*}$ is a cross term expressing interference

between $|u_n\rangle$ and $|u_m\rangle$

these appear when $|\psi_k\rangle$ is a coherent linear superposition of these states.

f_{nm} — weighted average of these terms taken over all possible states of the mixture

If $f_{nm} = 0 \Rightarrow$ the statistical average has cancelled out any interference effects between $|u_n\rangle$ and $|u_m\rangle$

If it is nonzero \Rightarrow certain coherence persists between $|u_n\rangle$ and $|u_m\rangle$

\Rightarrow off-diag. terms \equiv called coherences.

Important observations:

- * The distinction between 'population' and 'coherence' depends on the choice of basis $\{|u_n\rangle\}$

Since ρ is Hermitian: It is always possible to find an orthonormal basis $\{|x_n\rangle\}$ in which ρ is diagonal

$$\rho = \sum_l \pi_l |x_l\rangle \langle x_l| \Rightarrow \rho \text{ can thus be thought of as a statistical mixture of orthonormal states } |x_n\rangle \text{ with prob } \pi_n$$

there are no coherences between states $|x_n\rangle$



thought of as a statistical mixture of orthonormal states $|x_n\rangle$ with prob π_n

In such a mixture

$$\text{Tr } \rho^2 = \sum_l \pi_l^2 \leq \sum_l \pi_l = 1.$$

When one of the coefficients π_l equals 1, all others must be equal to zero.

In that case ρ is a pure state $\text{Tr } \rho^2 = 1$
for mixed st. $\text{Tr } \rho^2 < 1$

- * Assume the H to be time independent | Let $|u_n\rangle$ be the eigenvectors of H

$$\Rightarrow H |u_n\rangle = E_n |u_n\rangle$$

$$\langle u_n | H = \langle u_n | E_n$$

$$i\hbar \frac{d}{dt} \langle u_n | \rho | u_m \rangle = \langle u_n | (H\rho - \rho H) | u_m \rangle$$

$$= \langle u_n | E_n \rho - \rho E_m | u_m \rangle$$

$$= (E_n - E_m) \langle u_n | \rho | u_m \rangle$$

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$$i\hbar \frac{dp_{nm}}{dt} = (E_n - E_m) p_{nm} : \text{solution.}$$

$$p_{nn}(t) = \text{Const} \Leftarrow p_{nm}(t) = p_{nm}(0) e^{-\frac{i}{\hbar}(E_n - E_m)t}$$

and coherence oscillates at the Bohr freq.

$$\hbar(\omega_n - \omega_m) = E_n - E_m \Rightarrow \omega = \omega_n - \omega_m$$

(*)

$$p_{nn} p_{mm} \geq |p_{nm}|^2$$

$$\text{L.H.S} \Rightarrow \left(\sum_k p_k |c_n^{(k)}|^2 \right) \left(\sum_k p_k |c_m^{(k)}|^2 \right)$$

$$\geq \left(\sum_k p_k |c_n^{(k)} c_m^{(k)}| \right)^2 \geq \left| \sum_k p_k c_n^{(k)} c_m^{(k)} \right|^2 = |p_{nm}|^2$$

Consequence

\Rightarrow only p can have coherence between states whose populations are not zero

$$\text{Tr } p^2 = \sum_{mn} p_{mn} p_{nm} = \sum_{mn} |p_{mn}|^2 \leq \sum_{nm} p_{nn} p_{mm}$$

$$\boxed{\text{Tr } p^2 \leq 1.}$$

$$= \sum_n p_{nn} \sum_m p_{mm} = 1.$$

Ex 1. Let the density of stat. for a 2-level syst. be

$$p = \frac{1}{4} |a\rangle\langle a| + \frac{3}{4} |b\rangle\langle b|$$

? mixed or pure state

$$p^2 = \frac{1}{16} |a\rangle\langle a| + \frac{9}{16} |b\rangle\langle b|$$

$$\text{Tr } p^2 = \frac{10}{16} < 1$$

\Rightarrow mixed.

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Ex 2. Let $\rho = \frac{1}{4}|a\rangle\langle a| + \frac{3}{4}|b\rangle\langle b| + \frac{\sqrt{3}}{4}|a\rangle\langle b| + \frac{\sqrt{3}}{4}|b\rangle\langle a|$
 show that ρ — pure state.

$$\begin{pmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix}^2 = \begin{pmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix} \quad \rho^2 = \rho$$

$$\text{Tr}(\rho^2) = \text{Tr}(\rho) = 1.$$

Ex 3. Let system \mathcal{Q}_1 have $\rho_1 = \frac{1}{4}|a\rangle\langle a| + \frac{3}{4}|b\rangle\langle b|$
 $\mathcal{Q}_2 : \rho_2 = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| =$
 $= \frac{1}{2} \left(\frac{1}{\sqrt{4}}|a\rangle + \frac{\sqrt{3}}{\sqrt{4}}|b\rangle \right) \left(\langle a| \frac{1}{\sqrt{4}} + \langle b| \frac{\sqrt{3}}{\sqrt{4}} \right)$
 $+ \frac{1}{2} \left(\frac{1}{\sqrt{4}}|a\rangle - \frac{\sqrt{3}}{\sqrt{4}}|b\rangle \right) \left(\langle a| \frac{1}{\sqrt{4}} - \langle b| \frac{\sqrt{3}}{\sqrt{4}} \right) = \frac{1}{4}|a\rangle\langle a| + \frac{3}{4}|b\rangle\langle b|$
 $= \rho_1.$

\Rightarrow Two different realizations
 may correspond to the same
 statistical quantum ensemble.

Ex 3. \Uparrow Let $|a\rangle$ and $|b\rangle$ two orthonormal vectors
 Let \mathcal{Q}_1 be a quantum system prepared in
 state $|a\rangle$ with prob $\frac{1}{4}$ and in $|b\rangle$ with prob $\frac{3}{4}$

Let $|0\rangle = \frac{1}{\sqrt{4}}|a\rangle + \frac{\sqrt{3}}{\sqrt{4}}|b\rangle$

$|1\rangle = \frac{1}{\sqrt{4}}|a\rangle - \frac{\sqrt{3}}{\sqrt{4}}|b\rangle$

Let \mathcal{Q}_2 be prepared in $|0\rangle$ with prob $\frac{1}{2}$
 and in $|1\rangle$ " " $\frac{1}{2}$

show that \mathcal{Q}_1 and \mathcal{Q}_2 both correspond
 to the same density operator.

Note. Since $|a\rangle$ and $|b\rangle$ are orthogonal to each
 other they are eigenstates of the density operator
 $\rho = \rho_1 = \rho_2$. In contrast $|0\rangle$ and $|1\rangle$ are
 not orthogonal eigenstates of ρ .

Ex 4. Consider two arbitrary vectors $|A\rangle$ and $|B\rangle$
show $\text{Tr } |A\rangle\langle B| = \langle B|A\rangle$

$$\text{Tr } |A\rangle\langle B| = \sum_n \langle n|A\rangle\langle B|n\rangle = \sum_n \langle B|n\rangle\langle n|A\rangle = \langle B|A\rangle.$$

Applications of density operator

System in thermal equilibrium.

$$\rho = Z^{-1} e^{-\frac{H}{kT}} \quad \left| \begin{array}{l} \text{Density operator} \\ \text{as per Quant. Stat. Mech.} \end{array} \right.$$

H - Hamiltonian of the system.

$$Z = \text{Tr} \{ e^{-H/kT} \} \quad \left| \begin{array}{l} \text{Use basis vectors } |u_n\rangle \\ \text{of } H. \end{array} \right.$$

$$\rho_{nn} = \frac{\langle n | e^{-H/kT} | n \rangle}{Z}$$

$$\rho_{nm} = Z^{-1} \langle u_n | e^{-\frac{H}{kT}} | u_m \rangle = 0 \quad \text{for } n \neq m.$$

\Rightarrow At thermal equilibrium population of the stationary states are exponentially decreasing functions of energy.
Coherence between stationary states $\equiv 0$.

Description of the part of a system.

Reduced density operator.

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Interaction of 2-level atom with light

Total Hamiltonian $H = H_0 + H_1$

\downarrow free \rightarrow interaction part.

H_0 is the energy of the atom when it is free

$$H_0 = \hbar \omega_a |a\rangle\langle a| + \hbar \omega_b |b\rangle\langle b|$$

Denote $\sigma_{aa} = |a\rangle\langle a|$ $\sigma_{bb} = |b\rangle\langle b|$ \rightarrow describe populations (probabilities) of a and b.

$$\langle \sigma_{aa} \rangle = \text{Tr } \rho |a\rangle\langle a| = \langle a | \rho | a \rangle = f_{aa}$$

$$\langle \sigma_{bb} \rangle = f_{bb} \rightarrow \text{population of atom in a or b.}$$

Energy of the atom $\langle H \rangle = \hbar \omega_a |a\rangle\langle a| + \hbar \omega_b |b\rangle\langle b|$
 $= \hbar \omega_a f_{aa} + \hbar \omega_b f_{bb}$

Recall Pauli operator σ_z

$$\sigma_z = |a\rangle\langle a| - |b\rangle\langle b|$$

describes inversion of population of the atom.

$$\langle \sigma_z \rangle = f_{aa} - f_{bb}$$

population difference

$$|a\rangle\langle a| = \frac{1}{2} (1 + \sigma_z)$$

$$|b\rangle\langle b| = \frac{1}{2} (1 - \sigma_z)$$

$$H_0 = \frac{\hbar \omega_{ab}}{2} \sigma_z + \frac{\hbar (\omega_a + \omega_b)}{2} \quad (*)$$

$$\omega_{ab} = \omega_a - \omega_b$$

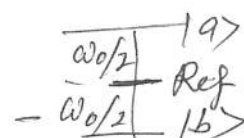
Transition frequency.

Choose ar. $\frac{\omega_a + \omega_b}{2}$ as energy reference

$$\Rightarrow \omega_a \Rightarrow \frac{\omega_0}{2}$$

$$\omega_b \Rightarrow -\frac{\omega_0}{2}$$

* Const = 0



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A constant in Hamiltonian can always be neglected since it commutes with the dens. operator and observables \Rightarrow does not affect the time evolution.

Interaction part

atom field interaction

Let the field be linearly polarized along x .
Use dipole approx \Rightarrow

$$H_i = -\vec{d} \cdot \vec{E} = -exE$$

$$= -e (|a\rangle\langle a| x_{aa} + |b\rangle\langle b| x_{bb} + |a\rangle\langle b| x_{ab} + |b\rangle\langle a| x_{ba}) E$$

$$= -e (|a\rangle\langle b| x_{ab} + |b\rangle\langle a| x_{ba}) E$$

$$= -(|a\rangle\langle b| x_{ab} + |b\rangle\langle a| x_{ba}) d_x E \quad x_{aa} = x_{bb} = 0$$

$$\text{Let } E = \frac{1}{2} (e^{-i\omega t} + e^{i\omega t}) E_0 \quad \frac{d_x = x_{ab} e}{\text{assume Real}}$$

$$\Rightarrow = E_0 \cos \omega t$$

$$\Rightarrow H_i = -\frac{\hbar}{2} \frac{d_x E_0}{\hbar} (e^{-i\omega t} + e^{i\omega t}) (|a\rangle\langle b| + |b\rangle\langle a|)$$

$$= -\frac{\hbar}{2} \Omega (e^{-i\omega t} + e^{i\omega t}) (|a\rangle\langle b| + |b\rangle\langle a|)$$

$$= -\frac{\hbar}{2} (\sigma_+ + \sigma_-) (e^{-i\omega t} + e^{i\omega t}) \Omega$$

$$\sigma_+ = \sigma_{ab} = |a\rangle\langle b|$$

$$\sigma_+ |a\rangle = 0$$

$$\sigma_- = \sigma_{ba} = |b\rangle\langle a|$$

$$\sigma_+ |b\rangle = |a\rangle$$

In Heisenberg picture $\sigma_+ = |a\rangle\langle b|$ $\sigma_- = |b\rangle\langle a|$
oscillate as $e^{i\omega_{ab}t}$ and $e^{-i\omega_{ab}t}$, respectively.

for a free atom $\Rightarrow \langle \sigma_- \rangle = \text{Tr} \rho |b\rangle\langle a| = \langle a | \rho | b \rangle = \rho_{ab}$
 $= \rho_{ab}(0) e^{-i\omega_{ab}t}$

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$|a\rangle\langle b|e^{i\omega t}$ and $|b\rangle\langle a|e^{-i\omega t}$
 vary quickly as $e^{\pm i(\omega + \omega_{ab})t}$

In contrast $|a\rangle\langle b|e^{-i\omega t}$ and $|b\rangle\langle a|e^{+i\omega t}$
 vary slowly as $e^{\pm i(\omega_{ab} - \omega)t}$

$$|a\rangle\langle b|e^{-i\omega t} \longrightarrow e^{+i(\omega_{ab} - \omega)t}$$

$$|b\rangle\langle a|e^{+i\omega t} \longrightarrow e^{-i(\omega_{ab} - \omega)t}$$

Coming back to the interaction Hamiltonian

$$\begin{aligned} H_1 &= -\frac{\hbar\Omega}{2} (\sigma_+ e^{-i\omega t} + \sigma_- e^{+i\omega t}) \\ &= -\frac{\hbar\Omega}{2} (|a\rangle\langle b|e^{-i\omega t} + |b\rangle\langle a|e^{+i\omega t}) \end{aligned}$$

Few properties of

$$\sigma_x = \sigma_+ + \sigma_-$$

$$\sigma_y = -i(\sigma_+ - \sigma_-)$$

$$\sigma_+^2 = \sigma_-^2 = 0$$

$$\sigma_z^2 = 1.$$

$$\sigma_+ = |a\rangle\langle b| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\sigma_- = |b\rangle\langle a| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_z = |a\rangle\langle a| - |b\rangle\langle b| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_+ \sigma_- = \frac{1}{2}(1 + \sigma_z) \quad \sigma_z \sigma_+ = \sigma_+$$

$$\sigma_- \sigma_+ = \frac{1}{2}(1 - \sigma_z) \quad \sigma_z \sigma_- = -\sigma_-$$

Commutation \Rightarrow any function $f(\sigma_+, \sigma_-, \sigma_z)$
 can also be written as
 a linear form.
 $c_+ \sigma_+ + c_- \sigma_- + c_z \sigma_z$

$$[\sigma_z, \sigma_+] = 2\sigma_+$$

$$[\sigma_z, \sigma_-] = -2\sigma_-$$

$$[\sigma_+, \sigma_-] = \sigma_z$$

①

Consider a free 2-level atom with $H = H_0 = \frac{\hbar\omega_0}{2} \sigma_z$
 Use commutation relations to write down eqns for σ 's
 $\sigma_z, \sigma_+, \sigma_-$

$$\frac{d\sigma_z}{dt} = \frac{i\omega_0}{2} [\sigma_z, \sigma_z] = 0$$

$$\frac{d\sigma_+}{dt} = \frac{i\omega_0}{2} [\sigma_z, \sigma_+] = i\omega_0 \sigma_+$$

$$\frac{d\sigma_-}{dt} = \frac{i\omega_0}{2} [\sigma_z, \sigma_-] = -i\omega_0 \sigma_-$$

$$\frac{dO}{dt} = \frac{i}{\hbar} [H, O]$$

$$\left. \begin{array}{l} \frac{d\sigma_+}{dt} = i\omega_0 \sigma_+ \\ \frac{d\sigma_-}{dt} = -i\omega_0 \sigma_- \end{array} \right\} \Rightarrow \begin{array}{l} \sigma_+ = \sigma_+(0) e^{i\omega_0 t} \\ \sigma_- = \sigma_-(0) e^{-i\omega_0 t} \end{array}$$

Evolution eqn of the density operator of a general quantum state of 2-level atom with no decay

$$\frac{d\rho}{dt} = \frac{i}{\hbar} [H, \rho]$$

$$\text{Let } O = \frac{H\rho}{\hbar} \Rightarrow \frac{d\rho}{dt} = -i(O - O^\dagger)$$

$$\frac{d}{dt} \rho_{\alpha\beta} = -i(\rho_{\alpha\beta} - \rho_{\alpha\beta}^\dagger) = -i(\rho_{\alpha\beta} - \rho_{\beta\alpha}^*)$$

Let us calculate $\rho_{\alpha\beta}$ $H = H_0 + H_1$

$$H = \hbar\omega_a |a\rangle\langle a| + \hbar\omega_b |b\rangle\langle b|$$

$$\omega_a = \frac{\omega_0}{2}$$

$$\omega_b = -\frac{\omega_0}{2}$$

$$O = \frac{H\rho}{\hbar} = \left(\omega_a |a\rangle\langle a| + \omega_b |b\rangle\langle b| \right) \rho - \frac{\Omega}{2} \left(|a\rangle\langle b| e^{-i\omega t} \rho + |b\rangle\langle a| e^{i\omega t} \rho \right)$$

$$\rho_{aa} = \omega_a \rho_{aa} - \frac{\Omega}{2} \rho_{ba} e^{+i\omega t}$$

$$\rho_{bb} = \omega_b \rho_{bb} - \frac{\Omega}{2} \rho_{ab} e^{-i\omega t}$$

$$\rho_{ab} = \omega_a \rho_{ab} - \frac{\Omega}{2} \rho_{bb} e^{i\omega t}$$

$$\rho_{ba} = \omega_b \rho_{ba} - \frac{\Omega}{2} \rho_{aa} e^{-i\omega t}$$

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$$\frac{d}{dt} \rho_{aa} = \frac{i\Omega}{2} (e^{-i\omega t} \rho_{ba} - e^{+i\omega t} \rho_{ab})$$

$$\frac{d}{dt} \rho_{bb} = -\frac{i\Omega}{2} (e^{-i\omega t} \rho_{ba} - e^{+i\omega t} \rho_{ab})$$

$$\frac{d}{dt} \rho_{ab} = -i\omega_{ab} \rho_{ab} - \frac{i\Omega}{2} e^{-i\omega t} (\rho_{aa} - \rho_{bb})$$

We can also define

$$\begin{aligned} \rho_{aa} &= c_a c_a^* & \rho_{ab} &= c_a c_b^* \\ \rho_{bb} &= c_b c_b^* & \rho_{ba} &= c_a^* c_b \end{aligned}$$

Introduce

$$\begin{aligned} \tilde{\rho}_{ab} &= \rho_{ab} e^{+i\omega t} \\ \tilde{\rho}_{ba} &= \rho_{ba} e^{-i\omega t} \\ \tilde{\rho}_{aa} &= \rho_{aa} \\ \tilde{\rho}_{bb} &= \rho_{bb} \end{aligned} \quad \left| \quad \frac{d}{dt} \tilde{\rho}_{ab} = \left(\frac{d\rho_{ab}}{dt} + i\omega \rho_{ab} \right) e^{i\omega t} \right.$$

$$\Rightarrow \left\{ \begin{aligned} \frac{d}{dt} \tilde{\rho}_{aa} &= \frac{i\Omega}{2} (\tilde{\rho}_{ba} - \tilde{\rho}_{ab}) \\ \frac{d}{dt} \tilde{\rho}_{bb} &= -\frac{i\Omega}{2} (\tilde{\rho}_{ba} - \tilde{\rho}_{ab}) \\ \frac{d}{dt} \tilde{\rho}_{ab} &= i\delta \tilde{\rho}_{ab} - \frac{i\Omega}{2} (\tilde{\rho}_{aa} - \tilde{\rho}_{bb}) \end{aligned} \right. \quad \delta =$$

Optical Bloch Eqs. without decay

Example. Consider a situation when the atom is initially in a statistical mixture of $|a\rangle$ and $|b\rangle$ with weight factor $\frac{1}{2} = \rho_a = \rho_b$. In this case the matrix elements of the initial density matrix

$$\begin{aligned} \rho_{aa}(0) &= \rho_{bb}(0) = \frac{1}{2} \\ \rho_{ab}(0) &= \rho_{ba}(0) = 0 \end{aligned}$$

Consequently $\tilde{\rho}_{aa} = \tilde{\rho}_{bb} = \frac{1}{2}$, $\tilde{\rho}_{ab} = \tilde{\rho}_{ba} = 0$ / at $t=0$ (3)
 Bloch Eqs. show that above values are true for
 any time t ; i.e.,

$$\begin{aligned} \tilde{\rho}_{aa}(t) &= \tilde{\rho}_{bb}(t) = \frac{1}{2} \\ \tilde{\rho}_{ab}(t) &= \tilde{\rho}_{ba}(t) = 0 \end{aligned} \quad \left| \quad \begin{array}{l} \text{As compared to} \\ \text{the previous case of} \\ \text{pure state this} \\ \text{gives the same} \\ \text{population but} \\ \text{diff. coherence.} \end{array} \right.$$

⊛ holds for pure and mixed case.
 though the derivation was for
 pure case.

In a system with decay.

$$\frac{d}{dt} \rho_{aa} = -\Gamma \rho_{aa} + \frac{i\Omega}{2} (e^{-i\omega t} \rho_{ba} - e^{i\omega t} \rho_{ab})$$

$$\frac{d}{dt} \rho_{bb} = -\Gamma \rho_{bb} - \frac{i\Omega}{2} (e^{-i\omega t} \rho_{ba} - e^{i\omega t} \rho_{ab})$$

$$\frac{d}{dt} \rho_{ab} = -\left(i\omega_{ab} + \frac{\Gamma}{2}\right) \rho_{ab} - \frac{i\Omega}{2} e^{-i\omega t} (\rho_{aa} - \rho_{bb}) + i\omega t$$

Again introducing

$$\tilde{\rho}_{ab} = \rho_{ab} e^{+i\omega t}$$

$$\tilde{\rho}_{ba} = \rho_{ba} e^{-i\omega t}$$

$$\tilde{\rho}_{aa} = \rho_{aa} , \tilde{\rho}_{bb} = \rho_{bb}$$

$$\frac{d}{dt} \tilde{\rho}_{aa} = -\Gamma \tilde{\rho}_{aa} + i\frac{\Omega}{2} (\tilde{\rho}_{ba} - \tilde{\rho}_{ab})$$

$$\frac{d}{dt} \tilde{\rho}_{bb} = -\Gamma \tilde{\rho}_{bb} - i\frac{\Omega}{2} (\tilde{\rho}_{ba} - \tilde{\rho}_{ab})$$

$$\frac{d}{dt} \tilde{\rho}_{ab} = \left(i\delta - \frac{\Gamma}{2}\right) \tilde{\rho}_{ab} - i\frac{\Omega}{2} (\tilde{\rho}_{aa} - \tilde{\rho}_{bb})$$