Gain and Dispersion.

Let us now discuss the influence of this suscept. on wave propagation.

Consider a plane wave propagating along z with complex amplitude

$$E = A e^{-i\omega t + ikz} + c.c.$$

This wave must obey the Helmholtz eqn. / wave eqn.

$$\frac{\partial^2 E}{\partial z^2} + \frac{\omega^2}{c_0^2}(1+\chi)E = 0 \quad \Big| \quad \frac{\partial^2 E}{\partial t^2} = \frac{1}{c_0^2}(1+\chi)\frac{\partial^2 E}{\partial t^2}$$

$$\frac{d^2 E}{dz^2} + k^2(E) = 0.$$

Here $\chi = \chi_{at} + \chi_{mat}$. total susceptib.

$$k = \frac{\omega}{c_0}(1+\chi)^{1/2}$$

$$\cong \frac{\omega n_0}{c_0}\left(1 + \frac{\chi'}{2n_0^2} + i\frac{\chi''}{2n_0^2}\right)$$

$n_0 = \sqrt{1+\chi_{mat}}$.

$$\frac{\omega}{c_0}\Big(\underbrace{1 + \chi_{mat}}_{= n_0^2} + \chi_{at}\Big)^{1/2}$$

$$= \frac{\omega n_0}{c_0}\left(1 + \frac{\chi_{at}}{n_0^2}\right)^{1/2}$$

$$= \frac{\omega n_0}{c_0}\left(1 + \frac{\chi_{at}}{2n_0^2}\right).$$

Then the expression for the plane wave reduces to

$$E = A e^{-i\omega t} \cdot \exp\left[i\frac{n_0\omega}{c_0}\left(1+\frac{\chi'}{2n_0^2}\right)z\right]$$

$$\times \exp\left[-\frac{n_0\omega_0}{c_0}\frac{\chi''_{at}}{2n_0^2}z\right] + c.c.$$

Wave propagation with a refractive index

$$n(\omega) = n_0\left(1 + \frac{\chi'}{2n_0^2}\right) \quad \text{(dispersion)}$$

with gain per unit length. $[\,I(z) = I(0)e^{\alpha z}\,]$

$$\alpha = -\frac{\omega\chi''}{n_0 c_0}$$

Recall that in steady state regime, for a plane wave propagating along $+z$, we have

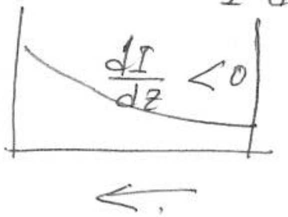
$$\alpha = \frac{1}{I}\frac{dI}{dz}$$

Light is amplified when $\alpha > 0$, i.e., when

$$\boxed{\chi''_{af} < 0} \Rightarrow \boxed{\rho_{22} > \rho_{11}}$$

Gain $> 0 \Leftrightarrow$ inversion of population.

* For a wave propagating in $-z$ dirn gain / unit length would be $\alpha = -\frac{1}{I}\frac{dI}{dz}$



Laser cross-section

The laser cross-section is defined in terms of gain coeff. per unit length as per the eqn

$$\alpha(\omega) = \sigma(\omega)\,\Delta n.$$

Δn population inversion / unit vol $\qquad \Delta n = n(\rho_{22} - \rho_{11}) = n_2 - n_1$

$n \rightarrow$ total population

$n_{1,2}$ — population in level 1 and 2.

$n\rho_{22} = n_2$

$n\rho_{11} = n_1$

$1 = \rho_{22} + \rho_{11} = \frac{n_2}{n} + \frac{n_1}{n} = 1$

Saturation intensity I_{sat} can be expressed as

$$I_{sat}(\omega) = \frac{\hbar\omega_0}{\sigma(\omega)\,\tau}$$

provided one defines the response time $\tau = \tau_2 + \tau_1 - A\tau_1\tau_2$

Thus $\quad \alpha(\omega) = \dfrac{\alpha_0(\omega)}{1 + I / I_{sat}(\omega)} = \sigma(\omega)\,\Delta n = \dfrac{\sigma(\omega)\,\Delta n_0}{1 + \dfrac{I}{I_{sat}(\omega)}}$

α_0 is the unsaturated gain coeff.

Δn_0 is the unsaturated population inversion

α and Δn — saturated gain and population.

Usefulness of crossection $\Longrightarrow$ it allows to make the difference between the characteristics of the transition (lie in the value of σ and its frequency dependence) and the status of population inversion.

— Generalization to degenerate levels.

Till now we considered simple 2-level system.

Let the levels 1 and 2 be degenerate

Let g_1 and g_2 — be their degeneracies.

Stimulated emission and absorption : result from summing over all possible transitions between different sublevels.

previous eqns can be generalized to

$$\Delta n = n_2 - \frac{g_2}{g_1} n_1 \quad \Big| \; n_2, n_1 \text{ populations in lev. 2 and 1.}$$

Generalization to more complicated profiles

We considered transitions whose width is due to finite lifetimes τ_1 and τ_2 of the levels and $\frac{1}{T'}$

Optical coherence $\frac{1}{T}$. In this case the transition exhibits a Lorentzian profile described by

$$\sigma(\omega) = \sigma(\omega_0)\, \frac{\pi \Delta\omega}{2}\, L(\omega-\omega_0)$$

Line centre lies at ω_0

normalized Lorentzian profile with FWHM $\Delta\omega$

$$L(\omega-\omega_0) = \frac{2}{\pi\Delta\omega}\, \frac{1}{1+\left[2\,\frac{\omega-\omega_0}{\Delta\omega}\right]^2}$$

The profile is normalized and hence has unit area.

$$\int_{-\infty}^{\infty} L(\omega-\omega_0)\, d\omega = 1.$$

All lineshapes are not Lorentzian

* When the transition comes from the addition of several transitions between nondegenerate sublevels, its profile can be very different.

Ex1: Happens in ions, experiencing Stark effect by the crystalline field due to the matrix in which they are embedded

In such cases, the normalized profile is called $g(\omega-\omega_0)$ and one has

$$\sigma(\omega) = \sigma(\omega_0)\frac{g(\omega-\omega_0)}{g(0)} = \sigma_0\, \frac{g(\omega-\omega_0)}{g(0)}.$$

Where σ_0 is the crosssection at the line centre with.

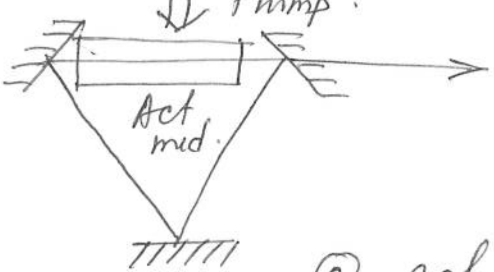
$$\int_{-\infty}^{\infty} g(\omega-\omega_0)\, d\omega = 1$$

Till now.

All the atoms same | $\Longrightarrow$ same crosssection $\Longrightarrow$ homog. broadened
| atoms - diff $\Longrightarrow$ inhomog. broadened.

Single mode Lasers.

Maxwell-Bloch Equations:



Unidirectional ring laser cavity.
Start with Bloch eqns, describing the atoms in the active medium.

(a) act. med. fills the whole cavity. | assumption.

(b) Intracavity field can be treated as a plane wave.

$$E(z,t) = A(z,t)e^{-i(\omega t - kz)} + c.c. = 2\mathrm{Re}\left[A(z,t)e^{-i(\omega t + kz)}\right]$$

(c) polarization is fixed $\Longrightarrow$ hence scalar expressi

(d) $A(z,t)$ slowly varying function of both z, t.

Equation for the evolution of polarization.

$$P_{at}(z,t) = (\mathcal{P}(z,t)e^{-i(\omega t - kz)} + c.c)$$
$$= 2\,\mathrm{Re}\left[\mathcal{P}(z,t)e^{-i(\omega t - kz)}\right]$$

$$\mathcal{P}(z,t) = n\,d\,\sigma_{21}(z,t).$$

$$\frac{d\sigma_{21}}{dt} = -(\Gamma - i\delta)\sigma_{21} - i\frac{\Omega}{2}(\rho_{22} - \rho_{11})$$

(x) nd. $\quad \Omega = \frac{dE_0}{\hbar} = \frac{2dA}{\hbar}$

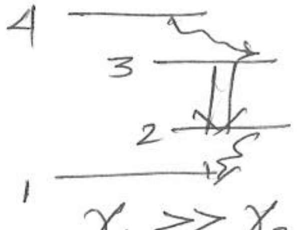
$$\Delta n = n(\rho_{22} - \rho_{11})$$

(*)
$$\boxed{\frac{d\mathcal{P}}{dt} = -(\Gamma - i\delta)\mathcal{P} - i\frac{d^2}{\hbar}\Delta n A}$$

Now. A is no longer given and liable to change.
atoms can change the field.

Evolution of population inversion.

In order to have a simpler description we restrict ourselves to '4-level' system



$\gamma_1 \gg \gamma_2$

We can assume that lower level is always empty $\rho_{11}=0$ $\Leftarrow$ | lower lasing level is not pumped: $\Lambda_1 \ll \Lambda_2$ | $\equiv$ ③ — ② — $\gamma_2 \gg \gamma_1$

Population inversion per unit vol } is given by $\Delta n = n_2 = \rho_{22} n$.

$$\frac{d\Delta n}{dt} = -\frac{\Delta n - \Delta n_0}{\tau} - \frac{i}{\hbar}\left(A^* P - A P^*\right) \Longleftarrow$$

$$\frac{d\rho_{22}}{dt} = \Lambda_2 - \gamma_2\rho_{22} - \frac{i}{2}\left(\Omega^*\sigma_{21} - \Omega\sigma_{21}^*\right)$$

$\boxed{\otimes n}$ $\quad \Omega = \frac{2Ad}{\hbar}$

$$\frac{d\Delta n}{dt} = -\frac{1}{\tau}\left(\Delta n - \Delta n_0\right) - \frac{i}{2}\left(\frac{2A^*d}{\hbar}\sigma_{21} - \frac{2Ad}{\hbar}\sigma_{21}^*\right)$$

We assumed $\tau = \tau_2 = \frac{1}{\gamma_2}$

Pumping term has been recast into $\frac{\Delta n_0}{\tau}$ $\quad \boxed{\Delta n_0 = n\Lambda_2\tau}$

Evolution of the field. $A(t, z)$.

E.M. field must be solution of Maxwell's eqns.

$$\frac{\partial^2 E}{\partial z^2} - \epsilon_0\mu_0\frac{\partial^2 E}{\partial t^2} - \sigma\mu_0\frac{\partial E}{\partial t} = \mu_0\frac{\partial^2 P}{\partial t^2}$$

overall average conductivity σ – uniformly distribut. in the cavity.

The total polarization $P = P_{mat} + P_{at}$

Contains the source term P_{at}

Which creates the el. m. field.

$\Longrightarrow$ it follows that atomic coherence is responsible for the emission of the laser field.

Replacing E and P by the corresponding slowly varying amplitudes,

$$\frac{c_0}{n_0}\frac{\partial A}{\partial z} + \frac{\partial A}{\partial t} + \frac{\sigma}{2\epsilon}A = i\frac{\omega}{2\epsilon}P.$$

ϵ - relative permittivity

Derivation:

$$P = P_{at} + \epsilon_0 \chi E, \quad P_{mat}$$

$$2ik\frac{\partial A}{\partial z} - k^2 A - \epsilon_0\mu_0\left(-2i\omega\frac{\partial A}{\partial t} - \omega^2 A\right) - \sigma\mu_0\left(-i\omega A\right)$$

$$= -\mu_0\omega^2 P_{at} + \mu_0\left(-2i\omega\epsilon_0\chi\frac{\partial A}{\partial t} - \omega^2\epsilon_0\chi A\right)$$

$$\Rightarrow 2ik\frac{\partial A}{\partial z} - k^2 A + 2i\omega\epsilon_0\mu_0(1+\chi)\frac{\partial A}{\partial t} + \omega^2\epsilon_0\mu_0(1+\chi)A + i\mu_0\omega\sigma A = -\mu_0\omega^2 P_{at}.$$

$$2ik\frac{\partial A}{\partial z} + 2i\omega\epsilon_0\mu_0\epsilon_r\frac{\partial A}{\partial t} - \left(k^2 - \omega^2\epsilon_0\mu_0\epsilon_r\right)A + i\mu_0\omega\sigma A = -\mu_0\omega^2 P_{at}$$

$$k^2 = \omega^2\epsilon_0\mu_0\epsilon_r = \frac{\omega^2}{c^2}\epsilon_r$$

$$\left(\frac{2i\frac{\omega}{c}n_0}{2i\omega\epsilon_0\mu_0\epsilon_r}\right)\frac{\partial A}{\partial z} + \frac{\partial A}{\partial t} + \frac{i\mu_0\omega\sigma}{2i\omega\epsilon_0\mu_0\epsilon_r}A = -\frac{\mu_0\omega^2 P_{at}}{2i\omega\epsilon_0\mu_0\epsilon_r}$$

$$\frac{1}{c_0^2}.$$

$$\boxed{\frac{c_0}{n_0}\frac{\partial A}{\partial z} + \frac{\partial A}{\partial t} + \frac{\sigma}{2\epsilon}A = i\frac{\omega}{2\epsilon}P_{at}}.$$

$\epsilon = \epsilon_0\epsilon_r$

$$\left|\frac{\partial^2 A}{\partial z^2}\right| \ll k\left|\frac{\partial A}{\partial z}\right| \qquad \left|\frac{\partial^2 P}{\partial t^2}\right| \ll \omega\left|\frac{\partial P}{\partial t}\right| \ll \omega\epsilon\left|\frac{\partial A}{\partial t}\right|$$

$$\left|\frac{\partial^2 A}{\partial t^2}\right| \ll \omega\left|\frac{\partial A}{\partial t}\right| \qquad \left|\frac{\partial A}{\partial t}\right| \ll \omega|A|$$

Further assume that the cavity has low loss in propagation

$\left|\frac{\partial A}{\partial z}\right| \ll 1$. $\Rightarrow$ we neglect $\underline{\underline{\frac{c_0}{n_0}\left|\frac{\partial A}{\partial z}\right| \ll \frac{\partial A}{\partial t}}}$.

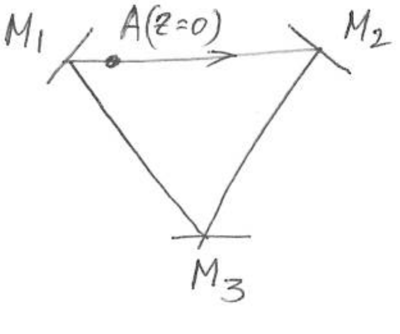
$$\frac{dA}{dt} = -\frac{1}{2\tau_{cav}} A + i\frac{\omega}{2\epsilon} \cancel{P_{at}}. \qquad \tau_{cav} = \frac{\epsilon}{\sigma}$$

lifetime of the photons in the cavity.

Finally we assume that the angular frequency ω is defined from the cavity resonance frequency ω_q by $\delta_{cav} = \omega - \omega_q$

$$\Longrightarrow \boxed{\frac{dA}{dt} = -\left(\frac{1}{\tau_{cav}} - i\delta_{cav}\right)A + i\frac{\omega}{2\epsilon} P}$$

<u>Cavity losses and detuning.</u>



$M_j \Leftrightarrow R_j, T_j$

Consider the intracav. field $A(z=0, t)$ at origin $A\,(z=0)$. After one round trip i.e. after time $\frac{L_{cav} \cdot n_0}{c_0}$ the field at pt. A is

$$A\left(z=0, t = t + \frac{L_{cav} n_0}{c_0}\right) = \sqrt{R_1 R_2 R_3 (1-\eta)}\, A(z=0, t)\, e^{ikL_{cav}}$$

R_j — intensity reflection coefficient.

η — other cavity losses. (diffusion, residual absorption from active med. and optical elements.

$$\left|A\left(z=0, t + \frac{L_{cav} n_0}{c_0}\right)\right| = \sqrt{R_1 R_2 R_3 (1-\eta)}\, \left|A(z=0, t)\right|$$

Since the round trip losses are weak.

$$\left| A\left(z=0,\ t+\frac{L_{cav}\, n_0}{c_0}\right)\right| = \left|A(z=0,t)\right| + \frac{L_{cav}\, n_0}{c_0}\cdot\frac{d|A|}{dt}.$$

$$\frac{d|A|}{dt} = -\left(\left|A(z=0,t)\right| - \left|A\left(z=0,\ t+\frac{L_{cav}n_0}{c_0}\right)\right|\right)\frac{c_0}{L_{cav}\, n_0}$$

$$= -\left(1 - \frac{\left|A\left(z=0,\ t=\frac{L_{cav}\, n_0}{c_0}+t\right)\right|}{\left|A(z=0,t)\right|}\right)\left|A(z=0,t)\right|\frac{c_0}{L_{cav}\, n_0}$$

$$\frac{d|A|}{dt} = -\left(1-\sqrt{R_1 R_2 R_3 (1-\eta)}\right)\frac{c_0}{L_{cav}\, n_0}\left|A(z=0,t)\right|$$

on the other hand.

$$\frac{d|A|}{dt} = -\frac{|A|}{2\tau_{cav}}.$$

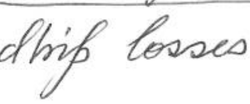
$$\Longrightarrow \quad \tau_{cav} = \frac{n_0 L_{cav}}{2c_0}\cdot\left(1-\sqrt{R_1R_2R_3(1-\eta)}\right)^{-1}.$$

$$1-R_j = T_j \ll 1.$$

$$\tau_{cav} \cong \frac{n_0 L_{cav}/c_0}{(1-R_1)+(1-R_2)+(1-R_3)+\eta} = \frac{\text{cav. roundtrip time}}{\text{roundtrip losses.}}$$

<u>Quality factor</u>:

$$Q_{cav} = \omega_0\,\frac{\text{stored energy}}{\text{dissipated power}} = \omega_0\,\frac{I}{-\frac{dI}{dt}} = \omega_0\tau_{cav}$$

In case of a Fabry-Perot cavity 

$L_{cav} = 2L$

twice the cavity length.