Steady state regime of a single freqn. laser

Steady state for ΔN and F

$$2^* = \begin{cases} 1 & \text{4-level} \\ 2 & \text{3-lev.} \end{cases}$$

Standard eqn. $\frac{dF}{dt} = -\frac{F}{\tau_{car}} + \varkappa F \Delta N$

$$\frac{d\Delta N}{dt} = -\frac{1}{\tau}(\Delta N - \Delta N_0) - 2^* \varkappa F \Delta N$$

$$\left.\begin{aligned} \frac{dF}{dt} &= 0 \\ \frac{d\Delta N}{dt} &= 0 \end{aligned}\right| \Rightarrow \text{off state } (F=0) \Rightarrow \boxed{\begin{aligned} F_{off} &= 0 \\ \Delta N_{off} &= \Delta N_0 \end{aligned}}$$

On-state: $\boxed{\Delta N_{on} = \frac{1}{\varkappa \tau_{car}} = \Delta N_{th}}$ $\Leftarrow$ $-\Delta N\left(\frac{1}{\tau_{car}} + \varkappa F\right) = -\frac{\Delta N_0}{\tau}$, $\Delta N = \frac{1}{\varkappa \tau_{car}}$

$$F_{on} = \frac{1}{2^* \varkappa \tau}\left(\frac{\Delta N_0}{\Delta N_{th}} - 1\right)$$

Def r – relative excitation.

$$\boxed{r = \frac{\Delta N_0}{\Delta N_{th}}} \Rightarrow \boxed{F_{on} = \frac{1}{2^* \varkappa \tau}(r-1)}$$

$$I_{on} = I_{sat}(r-1)$$

$$\Delta N_{th} = \frac{\Delta N_0}{1 + \frac{I}{I_{sat}}} \Leftarrow \frac{I}{I_{sat}} = r - 1 = \frac{\Delta N_0}{\Delta N_{th}} - 1.$$

$$\Delta N_{th} = \frac{\Delta N_0}{r} \uparrow \qquad \Rightarrow \cancel{\Delta N_0 = \Delta N_{th}}$$

$$r = 1 + \frac{I}{I_{sat}}$$

Stability of the off solution.

Denote the steady state solutions as $\bar{F}$ and $\overline{\Delta N}$

$$\begin{aligned} F(t) &= \bar{F} + x(t) \\ \Delta N(t) &= \overline{\Delta N} + y(t). \end{aligned}$$

Treat x, y as small quantities and linearize

Equation for photon no (2)

$$\frac{dF}{dt} = -\frac{F}{\tau_{cav}} + \kappa F \Delta N = -\frac{x}{\tau_{cav}} - \frac{\bar F}{\tau_{cav}} + \kappa(\bar F + x)(\overline{\Delta N} + y)$$

$$= \underbrace{\left(-\frac{\bar F}{\tau_{cav}} + \kappa \bar F \overline{\Delta N}\right)}_{\substack{\text{what we solved}\\ \text{for steady state}\\ \equiv 0}} - \frac{x}{\tau_{cav}} + \kappa \bar F y + \kappa \overline{\Delta N} x$$

dropped terms. xy quadratic ← small.

$$\boxed{\frac{dx}{dt} = + x\left(-\frac{1}{\tau_{cav}} + \kappa \overline{\Delta N}\right) + \kappa \bar F y}$$

Equation for ~~fie~~ inversion.

$$\frac{d\Delta N}{dt} = -\frac{1}{\tau}(\Delta N - \Delta N_0) - 2^* \kappa \Delta N F$$

$$= -\frac{1}{\tau}(\overline{\Delta N} - \Delta N_0) - \frac{y}{\tau} - 2^* \kappa \overline{\Delta N} \bar F - 2^* \kappa \bar F y - 2^* \kappa \overline{\Delta N} x$$

$$\frac{dy}{dt} = \underbrace{\left(-\frac{1}{\tau}(\overline{\Delta N} - \Delta N_0) - 2^* \kappa \overline{\Delta N} \bar F\right)}_{=0 \Rightarrow \text{steady st. eqn}} - 2^* \kappa \Delta N x - \left(\frac{1}{\tau} + 2^* \kappa \bar F\right) y$$

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \left(-\frac{1}{\tau_{cav}} + \kappa \overline{\Delta N}\right) & \kappa \bar F \\ -2^* \kappa \overline{\Delta N} & -\left(\frac{1}{\tau} + 2^* \kappa \bar F\right) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Eigenvalue equation

$$\left(\lambda + \frac{1}{\tau_{cav}} - \kappa \overline{\Delta N}\right)\left(\lambda + \frac{1}{\tau} + 2^* \kappa F\right) + 2^* \kappa^2 \bar F \Delta N = 0$$

Look at special cases of 'off' and 'on' solutions.

Stability of 'off' solution: $\overline{\Delta N} = \Delta N_0$
$\overline{F} = 0$.

Eigenvalue equation

$$\left(\lambda + \frac{1}{\tau_{cav}} - K\Delta N_0\right)\left(\lambda + \frac{1}{\tau}\right) = 0$$

$$\lambda_1 = -\frac{1}{\tau}, \quad \lambda_2 = -\frac{1}{\tau_{cav}} + K\Delta N_0$$

$$= K\left(\Delta N_0 - \frac{1}{K\tau_{cav}}\right).$$

$\Rightarrow$ λ_1 always negative (ok)
λ_2 becomes positive for $\Delta N_0 > \frac{1}{K\tau_{cav}}$
$= \Delta N_{th}$.

Thus below threshold:
Off solution is stable (Laser doesn't lase)
Beyond threshold 'off' solution becomes unstable (grows) $\longrightarrow$ into 'on' solution.

Stability of 'On' solution:

$$\overline{\Delta N} = \Delta N_{th} = \frac{1}{K\tau_{cav}}.$$

$$\overline{F} = \frac{1}{2^* K\tau}(r-1)$$

Eigenvalue eqn. becomes

$$\left(\lambda + \frac{1}{\tau_{cav}} - K\frac{1}{K\tau_{cav}}\right)\left(\lambda + \frac{1}{\tau} + 2^* K \frac{1}{2^* K\tau}(r-1)\right)$$

$$+ 2^* K^2 \frac{1}{2^* K\tau}(r-1)\frac{1}{K\tau_{cav}} = 0$$

$$\lambda\left(\lambda + \frac{r}{\tau}\right) + \frac{r-1}{\tau\tau_{cav}} = 0$$

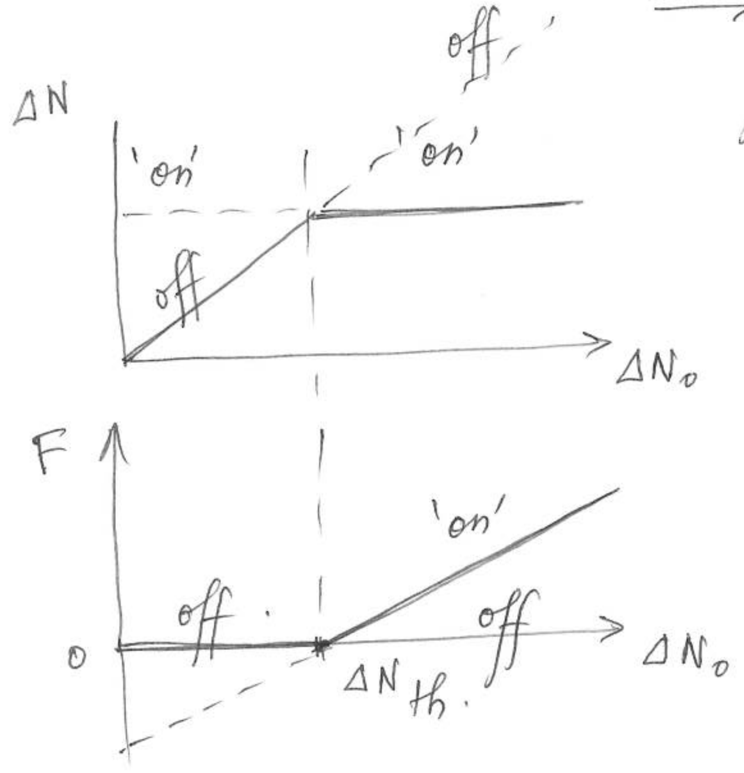
$$\boxed{\lambda_{1,2} = -\frac{r}{2\tau} \pm \frac{1}{2}\sqrt{\left(\frac{r}{\tau}\right)^2 - 4\frac{r-1}{\tau\tau_{cav}}}}$$

$\oplus$ root λ_1 with $\oplus$ sign becomes +ve for $r < 1$. $\quad r = \frac{\Delta N_0}{\Delta N_{th}}$.

$\Longrightarrow$ 'on' state will be unstable for unsaturated population inversion less than required for threshold. On $\longrightarrow$ off.

for $r > 1$. both the eigenvalues are negative (Real parts)

$\Longrightarrow$ stable on state for operation above threshold.

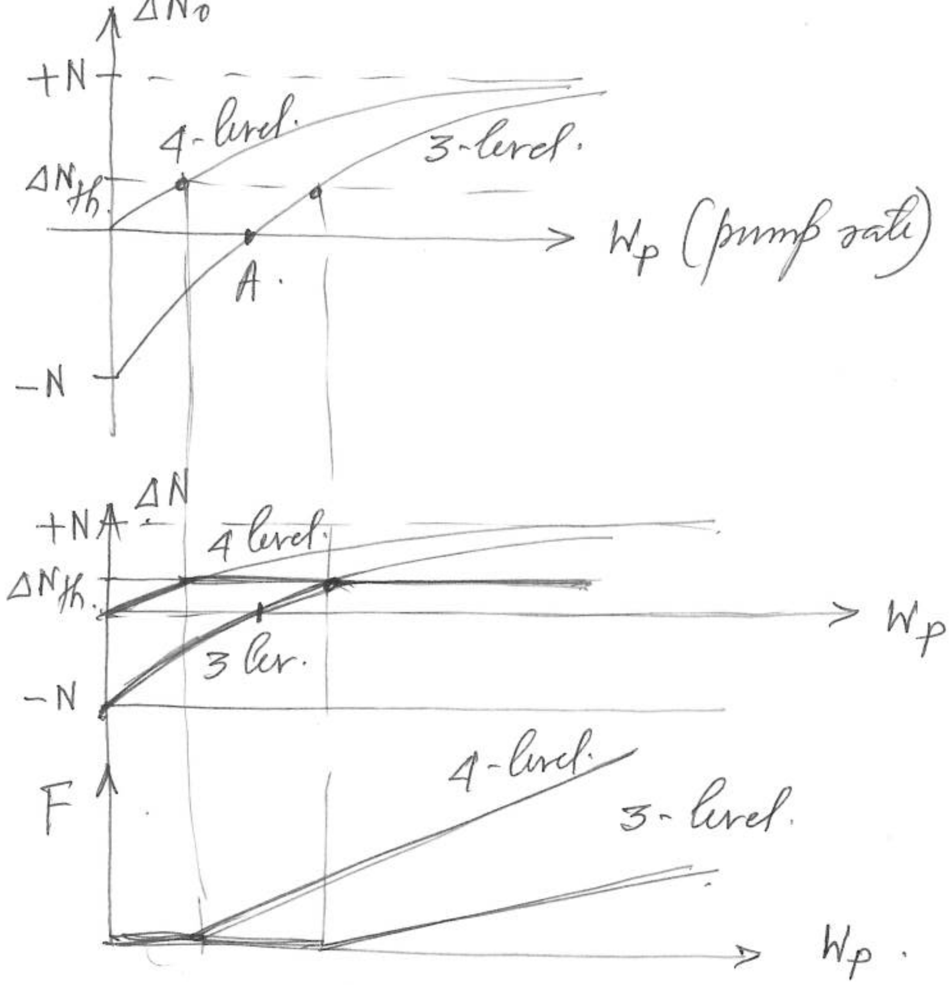


solid line – stable solution

dashed line – unstable solution

threshold actually corresponds to the transition from one steady state to the other steady state.

Further comparison of three and four level systems



Two disadvantages of 3-level system

① important amount of pumping goes to bleaching the medium. before gain can be obtained.

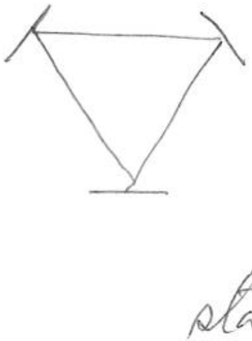
② I_{sat} for 3 level is two times smaller the slope of power vs pump is also twice smaller.

Laser frequency.

frequency pulling.

$\omega_L \rightarrow \omega$

Consider a laser cavity with active medium filling the cavity. Let the cavity be unidirectional so that



$$E(z,t) = A e^{-i\omega t + ikz} + c.c.$$

The fact that the laser is in steady state $\Longrightarrow$ The field is equal to itself after propagation through one roundtrip.

i.e., $\sqrt{R_1 R_2 R_3 (1-\eta)}\, e^{ikL_{cav}} = 1.$

In this eqn. k is complex

$$k = \frac{n_0 \omega}{c_0}\left(1 + \frac{\chi'_{at}}{2 n_0^2}\right) + i \frac{\omega}{c_0} \frac{\chi''_{at}}{2 n_0}$$

$$= \frac{n_0 \omega}{c_0}\left(1 + \frac{\chi'_{at}}{2 n_0^2}\right) - i \frac{\alpha}{2}.$$

Take modulus of the field.

$$\sqrt{R_1 R_2 R_3 (1-\eta)}\, e^{-k'' L_{cav}} = 1.$$

$$\ln\left[R_1 R_2 R_3 (1-\eta)\right] = \cancel{0} + 2 k'' L_{cav}$$

$$= -\alpha L_{cav}.$$

$\Longrightarrow$ $\alpha L_{cav} = \Pi$ $\Longrightarrow$ In steady state regime the saturated gain of the active medium exactly compensates the losses.

Frequency of oscillation.

On resonance.

$$\frac{n_0 \omega}{c_0}\left(1 + \frac{\chi'_{at}}{2 n_0^2}\right) L_{cav} = 2 q \pi.$$

Frequency ω_L of laser.

$$\boxed{\omega_L \left(1 + \frac{\chi'_{at}(\omega_L)}{2 n_0^2}\right) = 2\pi . q \frac{c_0}{n_0 L_{cav}}} = \omega_q$$

$\omega_q \rightarrow$ empty cavity resonance frequency.

$\Longrightarrow$ Active medium modifies the laser frequency with respect to the base cavity frequency ω_q

Let us suppose that the active medium has a Lorentzian gain with FWHM as $\Delta\omega_L$.

$$\chi'_{at}(\omega_L) = \frac{\omega_L - \omega_0}{\frac{\Delta\omega_L}{2}} \chi''(\omega_L) = \frac{\omega_L - \omega_0}{\frac{\Delta\omega_L}{2}} \frac{n_0 c_0}{\omega_0} \alpha(\omega_L)$$

Suppose now that ω_L is close to ω_q

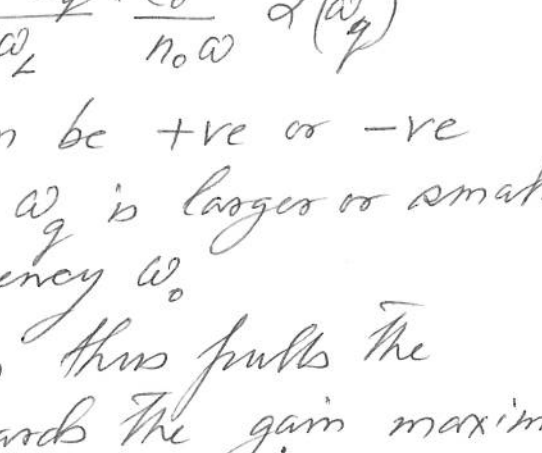
$\Longrightarrow$ the frequency shift created by the amplifying medium

$$\frac{\omega_L - \omega_q}{\omega_q} \cong \frac{\omega_0 - \omega_q}{\Delta\omega_L} \cdot \frac{c_0}{n_0 \omega} \alpha(\omega_q)$$

$\Longrightarrow$ the frequency shift can be +ve or −ve depending on whether ω_q is larger or smaller than the center frequency ω_0

$\Longrightarrow$ the active medium thus pulls the frequency ω_L towards the gain maximum

This is the so-called <u>frequency pulling</u> effect



In case when active med has length L_a and does not completely fill the cavity

$$\alpha L_a = \Pi$$

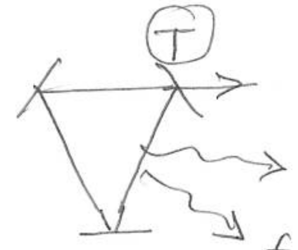
$$\frac{\omega_L - \omega_q}{\omega_q} \simeq \frac{\omega_0 - \omega_q}{\Delta\omega_L} \cdot \frac{c_0}{n_0 \omega} \frac{L_a \alpha(\omega_q)}{L_{cav.}}$$

Introduce the quantities

$$Q_{cav} = \omega_L \tau_{cav} = \frac{\omega}{c_0} \frac{L_{cav.opt}}{\Pi} \qquad Q_a = \frac{\omega_0}{\Delta\omega}$$

Laser power: Optimum output coupling

the steady state solution allows us to determine | numbers of photons and consequently the intensity inside the cavity.



One is interested in the output laser beam | goal: optimizing the output laser power by choosing the optimum output coupling.

extralosses γ.

Consider the laser, output coupling mirror has a transmission T.

All other losses (diffusion etc) correspond to amount of intensity loss per roundtrip

earlier solution

$$\gamma = \Pi - \mathbf{T}.$$

$$I = I_{sat}(r-1) = I_{sat}\left(\frac{\alpha_0 L_a}{T+\gamma} - 1\right).$$

$I_{on} = I_{sat}(r-1)$

L_a — length of the active medium.

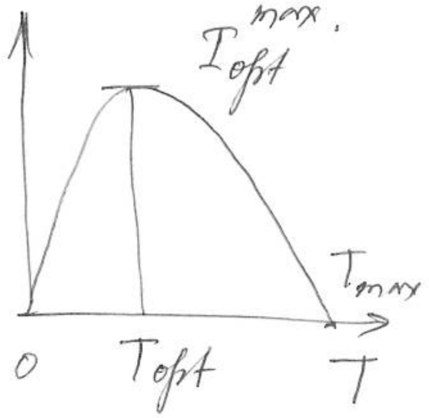
Output intensity

$$I_{out} = T I_{sat}\left(\frac{\alpha_0 L_a}{T+\gamma} - 1\right).$$

Plot of I_{out} vs. T.

Laser oscillates for $0 \leq T \leq T_{max} = \alpha_0 L_{a} - \gamma$

Optimal output power I^{max}_{out} at T_{opt}.



Order of magnitude of F_{sat}? ©

Example. Nd-YAG. $P = 100\,mW$ at $\lambda = 1.06\,\mu m$.

Let the cav.

$L_{car} = 20\,cm.$

$= 0.4\,m.$

photon flux $\phi = \frac{P}{\hbar c_0/\lambda}$

$= 5.3 \times 10^{16}$ ph/sec.

Noticeable cavity losses. 2% outcoupling.

thus $\tau_{car} = \frac{L_{car}}{c_0 \Pi} \sim 6.7 \times 10^{-8}$ sec.

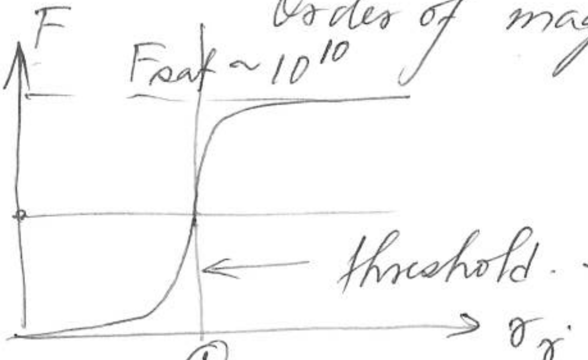
No of photons in the cavity

$$F = \tau_{car}\phi = 3.5 \times 10^{10}.$$

Far above threshold

$$F = F_{sat}(r-1)$$

Order of magnitude $F_{sat} \cong 10^{10}$



— corresponds to a dramatic increase in photon number.

(b)

$$T_{opt} = \sqrt{\alpha_0 L_a \gamma} - \gamma$$

$$I_{out}^{max} = I_{sat} \left(\sqrt{\alpha_0 L_a} - \sqrt{\gamma} \right)^2$$

In the limit when pumping is very strong and the laser is far above threshold. $\alpha_0 L_a \gg 1$

the maxm laser output power

$$\boxed{P_{out}^{max} = S\, I_{sat}\, \alpha_0 L_a = \frac{\hbar\omega \Delta N_0}{\tau}}$$

Maxm power that one can extract from the laser is equal to the energy that can be stored in the active medium, divided by gain recovery time τ.

Power in the vicinity of threshold.

Laser starts from spont. emission } laser when $r \sim 1$. neglecting spont emission not valid.

$$\frac{dF}{dt} = -\frac{F}{\tau_{cav}} + K \Delta N (F+1) \Delta N$$

$$= K \left[\Delta N (F+1) - \Delta N_{th} \right]$$

Steady state.

$$\frac{\Delta N}{\Delta N_{th}} = \frac{F}{F+1}$$

Let

$$r = \frac{\Delta N_0}{\Delta N_{th}}.$$

Besides, the saturation of the active med. reach.

$$\frac{\Delta N}{\Delta N_{th}} = \frac{F}{F+1}.$$

$$\frac{F}{F_{sat}}$$

$\Rightarrow$

$$\frac{\Delta N}{\Delta N_0} = \frac{1}{1 + F/F_{sat}}$$

$$\frac{F}{F_{sat}} = \frac{r-1}{2} \pm \sqrt{\left(\frac{r-1}{2}\right)^2 + \frac{r}{F_{sat}}}$$

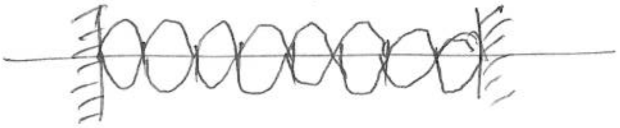
$$Q_{cav} = \omega \tau_{cav} = \frac{\omega}{c_0} \cdot \frac{L_{cav,\,opt}}{\Pi}$$

$$Q_a = \frac{\omega_0}{\Delta \omega} \qquad \omega_L = \frac{Q_{cav}\, \omega_q + Q_a\, \omega_0}{Q_{cav} + Q_{at}}.$$

Spatial hole-burning in a linear cavity

Till now intracav. mode → traveling plane wave.
Possible only in unidirectional ring cavity.

In a typical linear cavity → standing waves with nodes & anti-nodes



→ light energy density no longer homogeneous along prop. axis.

→ drastic modification of the saturation of the active medium and resulting lasing.

Standing waves. Saturation

$$\frac{\Delta n(z)}{\Delta n_0} = \frac{1}{1 + \frac{u_{standing}(z)}{u_{sat}}}$$

u – energy density.

$$u_{standing}(z) = 4\, u \cdot \sin^2\left(\frac{n_0 \omega}{c_0} z\right)$$

$$u_{sat} = \frac{n_0}{c_0} I_{sat} \qquad I = \frac{c_0}{n_0} u.$$

Output power.

Laser output power in presence of spatial hole burning.

adapt Eqn. $\dfrac{dF}{dt} = -\dfrac{F}{\tau_{cav}} + \varkappa F \Delta N$ to the case of spatial hole burning.

Variation due to gain of energy W stored inside.

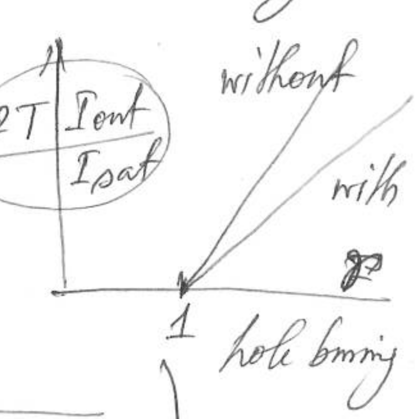
$$\frac{dW}{dt} = \frac{c_0}{n_0} S \sigma \int_0^{L_a} \frac{\Delta n_0 \, u_{standing}(z)}{1 + \frac{u_{standing}(z)}{u_{sat}}} dz$$

after integration

$$\left.\frac{dW}{dt}\right|_{gain} = \frac{c_0}{n_0} \sigma S \Delta n_0 \, u_{sat} L_a \left(1 - \frac{1}{\sqrt{1 + 4\frac{u}{u_{sat}}}}\right).$$

In steady state; this gain must exactly compensate the losses given by

$$\left.\frac{dW}{dt}\right|_{losses} = \frac{2 u V_{car}}{\tau_{car}}.$$



Combine. $\Longrightarrow$

$$\frac{u}{u_{sat}} = \frac{r}{2}\left(1 - \frac{1}{\sqrt{1 + 4\frac{u}{u_{sat}}}}\right).$$

third order polynom. Eqn. can be solved. $\Longrightarrow$

$$u = \frac{u_{sat}}{2}\left(r - \frac{1}{4} - \sqrt{\frac{r}{2} + \frac{1}{16}}\right)$$

$$I_{out} = \frac{T}{2} I_{sat}\left(r - \frac{1}{4} - \sqrt{\frac{r}{2} + \frac{1}{16}}\right)$$

must be compared with output intensity in absence of spatial hole burning

$$I_{out} = \frac{T}{2} I_{sat}(r-1)$$