Density matrix and Optical Bloch Eqns.

$|a\rangle$, $\hbar\omega_0$, $|b\rangle$

$$\dot\rho_{aa} = i\frac{\Omega}{2}\left(e^{-i\omega t}\rho_{ba} - e^{i\omega t}\rho_{ab}\right).$$

$$\dot\rho_{bb} = -\dot\rho_{aa}$$

$$\dot\rho_{ab} = -i\omega_0\rho_{ab} - i\frac{\Omega}{2}e^{-i\omega t}(\rho_{aa}-\rho_{bb}).$$

$|2\rangle$, ω_0, $|1\rangle$

$$\dot\rho_{11} = -\dot\rho_{22} = +i\frac{\Omega}{2}\left(\rho_{21}e^{i\omega t} - \rho_{12}e^{-i\omega t}\right)$$

$$\dot\rho_{21} = -i\omega_0\rho_{21} - i\frac{\Omega}{2}e^{-i\omega t}(\rho_{22}-\rho_{11})$$

$$\dot\rho_{12} = +i\omega_0\rho_{12} + i\frac{\Omega}{2}e^{i\omega t}(\rho_{22}-\rho_{11}).$$

Introduce: $\sigma_{12} = \rho_{12}e^{-i\omega t}$

$$\sigma_{21} = \rho_{21}e^{+i\omega t}$$

$$\frac{d\sigma_{21}}{dt} = \left(\dot\rho_{21} + i\omega\rho_{21}\right)e^{i\omega t} = \left(-i\omega_0\rho_{21} - i\frac{\Omega}{2}e^{-i\omega t}(\rho_{22}-\rho_{11}) + i\omega\rho_{21}\right)e^{i\omega t}$$

$$\boxed{\frac{d\sigma_{21}}{dt} = i\delta\sigma_{21} - i\frac{\Omega}{2}(\rho_{22}-\rho_{11})}$$

$$\boxed{\frac{d\rho_{11}}{dt} = i\frac{\Omega}{2}(\sigma_{21}-\sigma_{12})}$$

Introduce:

$u = \sigma_{21} + \sigma_{12}$.

$v = -i(\sigma_{12} - \sigma_{21})$

$w = \rho_{22} - \rho_{11}$

$$\frac{du}{dt} = \frac{d}{dt}(\sigma_{12}+\sigma_{21})$$

$$= i\delta(\sigma_{21}-\sigma_{12}) = -i(\sigma_{12}-\sigma_{21})\delta$$

$$\boxed{\dot u = \delta v}$$

Aux: $\dfrac{d\sigma_{12}}{dt} = -i\sigma_{12}\delta + i\dfrac{\Omega}{2}(\rho_{22}-\rho_{11})$

$$\frac{dv}{dt} = -i\frac{d}{dt}(\sigma_{12}-\sigma_{21}) = -i\left[-i\delta(\sigma_{12}+\sigma_{21}) + i\frac{\Omega}{\cancel{2}}(\rho_{22}-\rho_{11})\cancel{2}\right]$$

$$= -\delta u + \Omega w$$

$$\frac{dw}{dt} = -\Omega\, i\,(\sigma_{21}-\sigma_{12}) = -\Omega v.$$

$$\boxed{\begin{aligned} \dot{u} &= \delta v \\ \dot{v} &= -\delta u + \Omega w \\ \dot{w} &= -\Omega v \end{aligned}}$$

Let $\vec{R} = [u, v, w]$

$\vec{M} = [q_1, 0, q_3]$

$$\vec{M}\times\vec{R} = \begin{vmatrix} i & j & k \\ q_1 & 0_2 & q_3 \\ u & v & w \end{vmatrix}$$

$i \to \quad -vq_3 = \delta v \quad \boxed{q_3 = -\delta}$

$j \to \quad uq_3 - wq_1 = -\delta u + \Omega w$

$\boxed{q_3 = -\delta} \quad q_1 = -\Omega$

$k \to \quad q_1 v = -\Omega v \quad q_1 = -\Omega$

$\vec{M} = \underline{(-\Omega, 0, -\delta)}$

$$\circledast \quad \boxed{\frac{d\vec{R}}{dt} = \vec{M}\times\vec{R}}$$

<u>Discussion</u>: Role of $\vec{M}$ as a torque can easily be recognized from eqn. $\circledast$ since $\vec{M}\times\vec{R} \perp \vec{R}$ (Bloch vector)

The role of $\vec{M}$ is only to rotate $\vec{R}$ about the direction of $\vec{M}$, without changing its length.

$\vec{R}^2 = \vec{R}\cdot\vec{R}$ Constant

$$\frac{d\vec{R}^2}{dt} = 2\vec{R}\cdot\frac{d\vec{R}}{dt} = 2\vec{R}\cdot[\vec{M}\times\vec{R}] = 0$$

The constant magnitude of $\vec{R}$ has a physical meaning: $R^2 = u^2+v^2+w^2 = (\rho_{21}+\rho_{12})^2 - (\rho_{21}-\rho_{12})^2 + (\rho_{22}-\rho_{11})^2$

$$= 4\rho_{21}\rho_{12} + \rho_{22}^2 - 2\rho_{22}\rho_{11} + \rho_{11}^2$$

$$R^2 = 4\, c_2 c_1^* c_1 c_2^* + \left(c_2 c_2^*\right)^2 + 2\, c_2 c_2^* c_1 c_1^* + \left(c_1 c_1^*\right)^2$$
$$= \left(c_2 c_2^* + c_1 c_1^*\right)^2 = 1^2 = 1.$$

In absence of collisions and spont. emission the unit length of the Bloch vector is a statement for the conservation of probability in a 2-level syst.

Whenever $R^2 = 1$, the tip of the Bloch vector lies on the surface of the unit sphere.

Increasing and decreasing $\rho_{22} - \rho_{11} = w \Longrightarrow$ moving up or down in the vertical dirn.

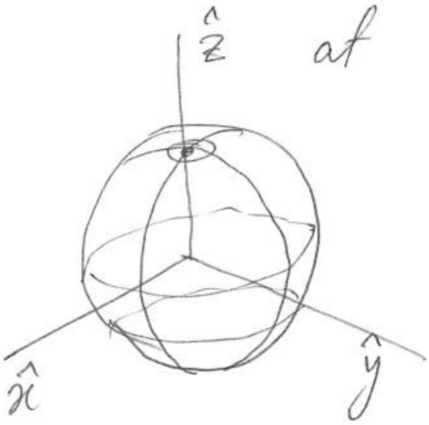
North pole: $\rho_{22} - \rho_{11} = 1$.
$\Longrightarrow \rho_{22} = 1$, $\rho_{11} = 0$.

south pole: $\rho_{22} = 0$ $\rho_{11} = 1$.
$w = -1$.
$u, v = 0$.

(N) — fully inverted 2-level system.

(S) — atom in the lower energy level.

Rotation angles of $\vec{R}$ have direct physical interpretation.



at resonance $\delta = \omega - \omega_0 = 0. \Longrightarrow$
dynamical evolution corresponds to rotation about the $\hat{x}$ axis.
since $\vec{M} = (-\Omega, 0, 0)$.

We can characterize the Bloch vector by a single parameter θ rotation angle about $\hat{x}$ axis

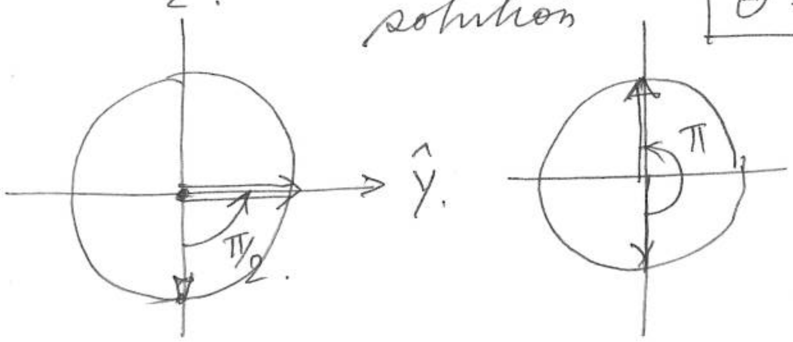
$v = -\sin\theta$
$w = -\cos\theta$

θ is measured from the south pole.

substitute in $\frac{dW}{dt} = -\Omega v$

$$+\frac{d\theta}{dt}\sin\theta = +\Omega\sin\theta \implies \boxed{\frac{d\theta}{dt} = \Omega}$$

solution $\boxed{\theta = \Omega t}$



Effect of $\pi/2$ and π pulse on on Bloch vector for an on resonance atom.

$$v = -\sin\Omega t$$
$$\rho_{22}-\rho_{11} = W = -\cos\Omega t$$

$\implies \rho_{11}+\rho_{22} = 1.$

$$\rho_{11} = \frac{1}{2}(1 - W)$$
$$\rho_{22} = \frac{1}{2}(1 + W)$$

$$\rho_{11} = \frac{1}{2}(1 + \cos\Omega t)$$
$$\rho_{22} = \frac{1}{2}(1 - \cos\Omega t)$$

Comment.

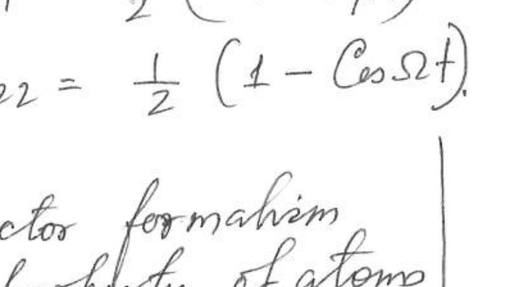
If the Rabi frequency is not constant

$$\theta(t) = \int_0^t \Omega(t')\,dt'$$
$$\Omega(t) = \frac{d\,E_0(t)}{\hbar}.$$

Bloch vector formalism connects a property of atoms (rotation angle on resonance) directly with the property of the incident radiation, namely, the time integral of the field amplitude.

$$\int \Omega(t')\,dt'$$

area of the pulse.



A π pulse turns θ through $180°$ about $\hat{x}$

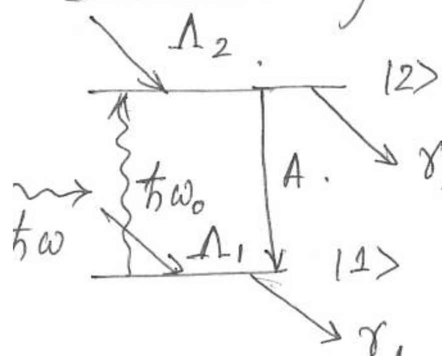
$\implies$ inverts atomic popul.

A 2π pulse turns the Bl. vector by $360°$ and returns it to the original state, same with 4π, 6π ...

$$\dot{\rho}_{11} = \frac{i\Omega}{2}\left(\sigma_{21} - \sigma_{12}\right) = -\dot{\rho}_{22}$$

$$\boxed{\begin{aligned} \frac{d}{dt}\left(\rho_{22} - \rho_{11}\right) &= -i\Omega\left(\sigma_{21} - \sigma_{12}\right) \\ \frac{d}{dt}\rho_{21} &= i\delta\,\sigma_{21} - i\frac{\Omega}{2}\left(\rho_{22} - \rho_{11}\right). \end{aligned}}$$

Inclusion of relaxation and pumping.



Laser transitions usually exhibit finite lifetimes τ_1, τ_2

Introduce relaxation rates

$$\gamma_1 = \frac{1}{\tau_1}, \quad \gamma_2 = \frac{1}{\tau_2} \quad \text{for } \rho_{11} \text{ and } \rho_{22}$$

Part of population decaying from $2 \to 1$ eg, by spont. emission. Let A be the associated decay rate.

(*) Coherence lifetime is usually much shorter than population lifetime.

Let Γ be the decay rate of coherence ρ_{12}.

One has $\Gamma \geq \frac{\gamma_1 + \gamma_2}{2}$.

$$\frac{d\rho_{22}}{dt} = \Lambda_2 - \gamma_2\rho_{22} - i\frac{\Omega}{2}\left(\sigma_{21} - \sigma_{12}\right).$$

$$\frac{d\rho_{11}}{dt} = \Lambda_1 - \gamma_1\rho_{11} + i\frac{\Omega}{2}\left(\sigma_{21} - \sigma_{12}\right) + A\rho_{22}$$

Compared to Fabien.
$\omega_1 = \omega_1^* = \Omega.$

$$\frac{d\sigma_{21}}{dt} = -\left(\Gamma - i\delta\right)\sigma_{21} - i\frac{\Omega}{2}\left(\rho_{22} - \rho_{11}\right).$$

Steady state.

Without the laser field $\Omega = 0$. Label the solution with superscript '0'

$$\sigma_{21}^{(0)} = \sigma_{12}^{(0)} = 0$$

$$\frac{\Lambda_2}{\gamma_2} = \rho_{22}^{(0)}$$

$$\frac{\Lambda_1}{\gamma_1} + \frac{A\rho_{22}^{(0)}}{\gamma_1} = \rho_{11}^{(0)}.$$

$$\frac{\Lambda_1}{\gamma_1} + A\frac{\Lambda_2}{\gamma_2\gamma_1} = \rho_{11}^{(0)}.$$

$$\rho_{22}^{(0)} - \rho_{11}^{(0)} = \frac{\Lambda_2}{\gamma_2} - \frac{\Lambda_1}{\gamma_1} - A\frac{\Lambda_2}{\gamma_2\gamma_1}$$

$$= \frac{\Lambda_2}{\gamma_2}\left(1 - \frac{A}{\gamma_1}\right) - \frac{\Lambda_1}{\gamma_1}.$$

$$= \frac{\Lambda_2(\gamma_1 - A) - \Lambda_1\gamma_2}{\gamma_1\gamma_2}$$

In order to get a popul. inversion. $\rho_{22} > \rho_{11}$ one must have.

$$\boxed{\gamma_1 > A}.$$

In this case inversion of population is obtained $\Leftarrow$ when $\Lambda_2\gamma_1 > \Lambda_1\gamma_2$ implying that upper level is pumped more efficiently than the lower level.

With Laser field. $\Omega \neq 0$.

$$\rho_{22} = \frac{\Lambda_2}{\gamma_2} - \frac{i\Omega}{2\gamma_2}(\sigma_{21} - \sigma_{12}) = \rho_{22}^{(0)} - \frac{i\Omega}{2\gamma_2}(\sigma_{21} - \sigma_{12})$$

$$\rho_{11} = \frac{\Lambda_1}{\gamma_1} + \frac{A}{\gamma_1}\underbrace{\left[\frac{\Lambda_2}{\gamma_2} - i\frac{\Omega}{2\gamma_2}(\sigma_{21} - \sigma_{12})\right]}_{\rho_{22}} + i\frac{\Omega}{2\gamma_1}(\sigma_{21} - \sigma_{12})$$

$$= \rho_{11}^{(0)} - i\frac{A\Omega}{2\gamma_1\gamma_2}(\sigma_{21} - \sigma_{12}) + i\frac{\Omega}{2\gamma_1}(\sigma_{21} - \sigma_{12})$$

$$\rho_{11} = \rho_{11}^{(0)} + i\frac{\Omega}{2\gamma_1\gamma_2}(\gamma_2 - A)(\sigma_{21} - \sigma_{12})$$

$$\boxed{\rho_{22} - \rho_{11} = \left(\rho_{22}^{(0)} - \rho_{11}^{(0)}\right) - \frac{i\Omega}{2\gamma_2\gamma_1}(\cancel{\gamma_1} + \gamma_2 - A)(\sigma_{21} - \sigma_{12})}$$

Eq.3 $\rightarrow$ $$\sigma_{21} = \frac{-i\frac{\Omega}{2}(\rho_{22}-\rho_{11})}{\Gamma - i\delta} = \frac{\frac{\Omega}{2}(\rho_{22}-\rho_{11})}{\delta + i\Gamma}$$ (7)

$$\sigma_{21} = \frac{\Omega}{2}\cdot(\rho_{22}-\rho_{11})\cdot\frac{\delta - i\Gamma}{\delta^2+\Gamma^2}$$

$$\sigma_{12} = \frac{\Omega}{2}\cdot(\rho_{22}-\rho_{11})\,\frac{\delta + i\Gamma}{\delta^2+\Gamma^2}$$

$$(\sigma_{21}-\sigma_{12}) = \frac{\Omega}{2}\cdot(\rho_{22}-\rho_{11})\frac{(-2i\Gamma)}{\delta^2+\Gamma^2} = -i\frac{\cancel{2}\Omega\Gamma}{\delta^2+\Gamma^2}(\rho_{22}-\rho_{11})$$

$$(\rho_{22}-\rho_{11}) = (\rho_{22}^{(0)}-\rho_{11}^{(0)}) - i\Omega\cdot\frac{(\gamma_1+\gamma_2-A)}{2\gamma_1\gamma_2}\cdot(-i)\frac{\Omega\Gamma}{\delta^2+\Gamma^2}(\rho_{22}-\rho_{11})$$

$$= (\rho_{22}^{(0)}-\rho_{11}^{(0)}) - \frac{\Omega^2(\gamma_1+\gamma_2-A)\,\Gamma}{2\gamma_1\gamma_2(\delta^2+\Gamma^2)}(\rho_{22}-\rho_{11})$$

$$\Longrightarrow \boxed{(\rho_{22}-\rho_{11}) = \frac{\rho_{22}^{(0)}-\rho_{11}^{(0)}}{\left(1 + \frac{\gamma_1+\gamma_2-A}{\gamma_1\gamma_2}\cdot\frac{\Omega^2}{2}\cdot\frac{\Gamma}{\delta^2+\Gamma^2}\right)}}$$

Lorentzian factor

Susceptibility

Saturation

$$\vec{E} = \hat{e}E_0 e^{-i\omega t} + c.c.$$

err $E_0 \rightarrow \frac{E_0}{2}$

$$u = \frac{1}{2}(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H})$$

$$\vec{S} = \vec{E}\times\vec{H}$$

$$I = 2\epsilon_0 n_0 c_0 |E_0|^2 \quad 4$$

$$\Omega^2 = \frac{d^2E_0^2}{\hbar^2} = \frac{d^2 I}{\hbar^2\, 2\epsilon_0 n_0 c_0} \quad 4$$

Introduce I_{sat}

$$I_{sat} = \frac{\epsilon_0 c_0 n_0 \hbar^2}{d^2\,\Gamma}\,\frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2-A}\cdot(\Gamma^2+\delta^2)$$

$$\rho_{22}-\rho_{11} = \frac{\rho_{22}^{(0)}-\rho_{11}^{(0)}}{1+\frac{I}{I_{sat}}}$$

When the intensity becomes much stronger than I_{sat}, the system becomes transparent and no longer responds to incident wave.

The incident wave creates a polarization in the atomic medium. In steady state, the response of the medium can be given by the susceptibility

Polarization: $P_{at,x} = n\langle d_x\rangle = n\,Tr\{\rho\hat{d}_x\}$

$$= 2nd\,Re(\rho_{21})$$

$$\begin{pmatrix}\rho_{11} & \rho_{12}\\ \rho_{21} & \rho_{22}\end{pmatrix}\begin{pmatrix}0 & d_{12}\\ d_{21} & 0\end{pmatrix}. \quad d = d_{12} = d_{21}.$$

Here n - atomic density

$$= \rho_{12}d_{21} + \rho_{21}d_{12} = 2d\,Re(\rho_{21})$$

$P_{at,x}$ — just the part of the polarization of the active medium, that comes from the active atom.

The polarization of the matrix in which the active atoms are embedded is given by

$$P_{mat,x} = \epsilon_0\chi_{mat}\frac{E_0}{2}e^{-i\omega t} + c.c.$$

Fabien: $\epsilon_0\chi A e^{-i\omega t} + c.c.$

Refractive index

$$n_0 = \sqrt{1+\chi_{mat}}.$$

Total polarization of the active medium

$$P_x = P_{mat,x} + P_{at,x}.$$

$$P_{at,x} = 2nd\,Re(\rho_{21})$$

$$= 2nd\,Re(\sigma_{21}e^{-i\omega t}) = \epsilon_0\chi_{at}\frac{E_0}{2}e^{-i\omega t} + c.c.$$

$$\Rightarrow nd\cdot\sigma_{21}e^{-i\omega t} + c.c. = \epsilon_0\chi_{at}\frac{E_0}{2}e^{-i\omega t} + c.c.$$

$$\boxed{\chi_{at} = \chi'_{at} + i\chi''_{at}} \qquad \boxed{\chi_{at} = \frac{2nd\sigma_{21}}{E_0\epsilon_0}}$$

Recall the expression for $\sigma_{21} = \frac{\Omega}{2}(\rho_{22} - \rho_{11})\frac{\delta - i\Gamma}{\delta^2 + \Gamma^2}$

$$\chi_{at} = \chi' + i\chi'' = \frac{2nd\Omega}{E_0\epsilon_0 \cancel{2}\cancel{\hbar}}\cdot(\rho_{22} - \rho_{11})\frac{\delta - i\Gamma}{\delta^2 + \Gamma^2}$$

$$= \frac{\cancel{2}nd^2E_0}{E_0\epsilon_0\hbar}(\rho_{22} - \rho_{11})\frac{\delta - i\Gamma}{\delta^2 + \Gamma^2}$$

$$(\rho_{22} - \rho_{11}) = \frac{\rho_{22}^{(0)} - \rho_{11}^{(0)}}{1 + \frac{I}{I_{sat}}}$$

Thus:

$$\chi'_{at}(\delta) = \frac{nd^2}{\epsilon_0\hbar}\,\frac{\rho_{22}^{(0)} - \rho_{11}^{(0)}}{1 + \frac{I}{I_{sat}}}\,\frac{\delta}{\delta^2 + \Gamma^2}$$

$$\chi''_{at}(\delta) = - \frac{nd^2}{\epsilon_0\hbar}\,\frac{\rho_{22}^{(0)} - \rho_{11}^{(0)}}{1 + \frac{I}{I_{sat}}}\cdot\frac{\Gamma}{\delta^2 + \Gamma^2}$$

In presence of inversion $\rho_{22}^{(0)} - \rho_{11}^{(0)} > 0$.

$\chi''_{at}(\delta) > 0 \Longrightarrow$ gain.

Else. $\chi''_{at}(\omega) < 0 \Longrightarrow$ loss when no inversion is there.

$\chi''_{at}(\omega) = 0$ when $\underline{\rho_{22}^{(0)} = \rho_{11}^{(0)}}$. — Saturation: the atomic medium neither absorbs nor emits.

χ'' — exhibits a Lorentzian line shape

χ' — gives the associated dispersion.

$$\chi'(\delta) = -\frac{\delta}{\Gamma}\chi''(\delta)$$

Note that $\frac{1}{1 + \frac{I}{I_{sat}}}\cdot\frac{1}{\Gamma^2 + \delta^2} = \frac{1}{\delta^2 + \Gamma^2\left(1 + \frac{I}{I_{sat}(\delta = 0)}\right)}$.

Saturation broadens the Lorentzian by a factor.

broadened by a factor

$$\sqrt{1+\frac{I}{I_{sat}(\delta=0)}}$$

This is the so-called saturation broadening or power broadening.

When I/I_{sat} increases $\Rightarrow$ the real and imagin. parts of susceptibility decrease and $\to 0$

At stronger intensities atoms no longer interact with the field!

Note also that

$$I_{sat}(\delta) = I_{sat}(\delta=0)\,\frac{\Gamma^2+\delta^2}{\Gamma^2}$$

With increasing detuning, interaction of the atoms with field gets less efficient.

Indeed:

$$I_{sat}(\delta) = \frac{\epsilon_0 c_0 n_0 \hbar^2}{d^2\Gamma}\,\frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2-A}\,(\Gamma^2+\delta^2)$$

$$I_{sat}(\delta=0) = \frac{\epsilon_0 c_0 n_0 \hbar^2}{d^2\Gamma}\cdot\frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2-A}\,\Gamma^2$$