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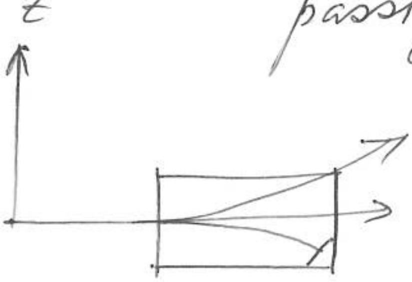
Spin states and density matrix

Pure spin states: Stern Gerlach Expt.

Consider a beam of spin $\frac{1}{2}$ particles (hydrogen) passing through a SG setup.



field gradient along z w.r.t. fixed coord. sys.

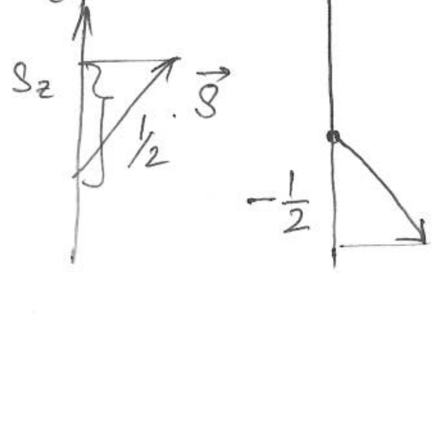
beam split vertically into two each correspond to one of the two possible eigenvalues of the compon. S_z of the spin operator $\vec{S}$ ($m = \pm\frac{1}{2}$)

one of the beams is stopped. (eliminated)

$\Rightarrow$ emerging particles are in a state, which corresponds to only one of the eigenvalues here it is $+\frac{1}{2}$

If the field gradient points in z' dirn. The quantum particles will be described by quant. no z'

$\Rightarrow$ If the incident beam contains only $+\frac{1}{2}$ spins. The beam will pass through completely



In all other cases part of the beam will be blocked off $\Rightarrow$ emerging beam will be less intense.

However, by tilting the apparatus at various angles about z. it may be possible to find an orientation allowing the whole beam.

$\Rightarrow$ all the particles behave in the same way $\Rightarrow$ deflected in the same fashion.

⊛ If it is possible to find an orientation of SG apparatus for which a given beam is completely transmitted, then we say that the beam is in a pure spin state.

If the state of a given beam is known to be pure then the joint state of all particles can be represented in terms of one and the same state vector $|\chi\rangle$

* important point

$m = +\frac{1}{2} \quad |\frac{1}{2}\rangle$

$m = -\frac{1}{2} \quad |-\frac{1}{2}\rangle$

(*) $|+\frac{1}{2}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ adj $(1, 0)$

$|-\frac{1}{2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (0, 1)$

If SG magnet is along z'

$$|\chi\rangle = |\pm\tfrac{1}{2}, z'\rangle$$

A general spin state $|\chi\rangle$ can always be written as

$$|\chi\rangle = a_1 |+\tfrac{1}{2}\rangle + a_2 |-\tfrac{1}{2}\rangle$$

In representation (*) $|\chi\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad \langle\chi| = (a_1^* \ a_2^*)$

the state $|\chi\rangle$ is normalized $\Rightarrow |a_1|^2 + |a_2|^2 = 1$
$= \langle\chi|\chi\rangle$

A pure spin state can be characterized either by specifying the polar angles or by (a_1, a_2)

<u>Example</u>: Polarization vector $\vec{P}$

$P_i = \langle\sigma_i\rangle$ expectation value of the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\langle\sigma_i\rangle = \langle\chi|\sigma_i|\chi\rangle$$

$P_x = (1, 0)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$ for a beam of particles in state $|+\frac{1}{2}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$P_y = (1, 0)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \qquad P_x^2 + P_y^2 + P_z^2 = 1$

$P_z \quad (1, 0)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = +1 \qquad |\pm\frac{1}{2}\rangle$ states of opposite polarization.

Consider now the general pure state $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

Let $\begin{cases} a_1 = \cos\frac{\theta}{2} \\ a_2 = \sin\frac{\theta}{2}\, e^{i\delta} \end{cases}$ $\longrightarrow$ | δ is the relative phase. Completely specified by two real numbers.

$$\circledast \quad |\chi\rangle = \begin{pmatrix} \cos\theta/2 \\ e^{i\delta}\sin\theta/2 \end{pmatrix}. \qquad P_x = \left(\cos\theta/2,\ e^{-i\delta}\sin\theta/2\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta/2 \\ e^{i\delta}\sin\theta/2 \end{pmatrix}.$$

$$P_x^2 + P_y^2 + P_z^2 = 1. \qquad \circledast \quad \begin{cases} P_x = \sin\theta\cos\delta \\ P_y = \sin\theta\sin\delta \\ P_z = \cos\theta \end{cases}$$

$\theta \to$ polar angle.
δ – azimuthal angle.

A second coord. syst x', y', z' can be chosen such that z' axis is parallel to $\vec{P}$. Taking z' as quantization axis

$$P_{x'} = 0, \quad P_{y'} = 0, \quad P_{z'} = 1.$$

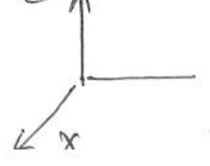
$\Longrightarrow$ all particles have spin up with respect to z'

$\Longrightarrow$ The direction of the pol. vector is the dirn along which all spins are pointing.

SG apparatus pointing along $\vec{P}$ will allow all the particles to pass through.

$\circledast$ and $\circledast$ allow explicit spin functions to be const.

Ex: Pure state with spin pointing along x



$|+\frac{1}{2}, x\rangle \qquad |\chi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\theta = 90^\circ, \ \delta = 0$

$-x'$ direction $|\chi\rangle = |\tfrac{1}{2}, +x\rangle$

$\theta = 90^\circ,\ \delta = 180^\circ \quad |\chi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

$|\chi\rangle = |\tfrac{1}{2}, y\rangle \quad \theta = 90,\ \delta = 90$

$= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} \qquad |-\tfrac{1}{2}, y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$

Note that. These four states are constructed using the superposition $|+\tfrac{1}{2}\rangle$ and $|-\tfrac{1}{2}\rangle$ states. using same magnitudes $|a_1| = |a_2| = \frac{1}{\sqrt{2}}$ but with different relative phases.

Mixed spin states.

most general spin states. for an ensemble of particles.

Propose two beams of particles. independently

one in pure $|+\tfrac{1}{2}\rangle$
other in pure $|-\tfrac{1}{2}\rangle$ | states.

independent : no definite phase relation exist between the two.

def in the first beam N_1 part
second beam N_2 part | Investigate the polarization state of the combined beam by a SG filter. for various orientat.

It is not possible to find any orientation $\Leftarrow$ for which the combined beam passes through completely.

$\Longrightarrow$ the joint beam is not in a pure state.

def: States which are not pure are called mixed states or mixtures.

How to describe such a mixed state?

(1). It is not possible to describe it by just one state vector $|\chi\rangle$ $\Longrightarrow$ since associated with this state there is a dirn along which all spins point.

$\equiv$ dirn of the pol. vector.

whole beam ~~wh~~ would have passed thro' a SG. apparatus.

(2) Can not be represented by a linear superposition of $|+\frac{1}{2}\rangle$ and $|-\frac{1}{2}\rangle$ of the two constituent beams.

for such a superposition, we need to know magnitudes and relative phases. δ

a_1, a_2

$$\left.\begin{array}{l} |a_1|^2 = W_1 \\ |a_2|^2 = W_2 \end{array}\right|$$ probabilities of finding the particles in the states $|+\frac{1}{2}\rangle$ or $|-\frac{1}{2}\rangle$ respectively.

$$W_1 = \frac{N_1}{N}, \quad W_2 \frac{N_2}{N} \qquad N = N_1 + N_2 \Longrightarrow W_1 + W_2 = 1$$

independently prepared $\Longrightarrow$ no def phase relation

N_1 particles prepared in state $|+\frac{1}{2}\rangle$

N_2 " " " $|-\frac{1}{2}\rangle$

mixture to be prepared relaining maxm. information.

$\vec{P}$ of the total beam is to be determined by taking the statistical average over the separate beams.

$$P_i = W_1 \langle \tfrac{1}{2} | \sigma_i | \tfrac{1}{2} \rangle + W_2 \langle -\tfrac{1}{2} | \sigma_i | -\tfrac{1}{2} \rangle$$

$$P_x = 0 \quad P_y = 0 \qquad P_z = W_1 - W_2 = \frac{N_1 - N_2}{N}$$

$$0 \leq |\vec{P}| \leq 1.$$

General treatment

Consider a quant. system denoted by $|\psi\rangle$ | pure state.
we have complete info; state vector

Often we do not know the exact state vector $\Rightarrow$ incomplete information.

Ex 1. photon emitted by a source of natural light
$\Rightarrow$ can have any polarization state.
with equal probability.

Ex. 2 a system in thermal equilibrium at T.
has a prob. $\sim e^{-\frac{E_n}{kT}}$ of being in state E_n.

"Incomplete information" $\equiv$ system may be either in
state $|\psi_1\rangle$ with prob p_1
or in $|\psi_2\rangle$ " " p_2
$\vdots$ $|\psi_n\rangle$ . . . . . p_n.

$\Rightarrow$ We have a statistical mixture of states
$|\psi_1\rangle, |\psi_2\rangle \ldots$ with probabilities $p_1, p_2 \ldots$

Def. A statistical mixture of two or more diff states is called a mixed state | of course.
$p_1 + p_2 \ldots = \sum_k p_k = 1$

A statistical mixture should not be confused with a system with $|\psi\rangle$ as a linear superposition

$$|\psi\rangle = \sum_k c_k |\psi_k\rangle \Rightarrow \text{has } \underline{\text{interference}}.$$

$$|\text{ampl}|^2 \sim c_k c_{k'}^*$$

for mixed state as stat mixture of $|\psi_k\rangle$'s
no interference.

<u>Example</u>. single particle in coord. sp.

when the particle is in a linear superpos. state

$$\psi(r) = \sum_k c_k \psi_k(r).$$

prob. of finding the particle at r $\Rightarrow |\psi(r)|^2 = |\sum_k c_k \psi_k(r)|^2 = P(r)$

$$= \sum_{kk'} \underbrace{c_k c_{k'}^* \psi_k \psi_{k'}^*}_{\text{interference}}$$

In a statistical mixture

$$P(r) = \sum_k p_k |\psi_k(r)|^2 \Rightarrow \text{no interference}.$$

<u>Density operator for pure states</u>.

Let the state vector of the system be perfectly known

$\Rightarrow$ all prob p_k's $= 0$ except one.

$\Rightarrow$ system is said to be in a pure state.

<u>Description of state vector</u>

$$\circledast \quad |\psi\rangle = \sum_n c_n |u_n\rangle \qquad \{|u_n\rangle\}$$

forms an orthonorm basis.

$\langle u_n | u_m\rangle = \delta_{nm}$.

In coord. represent.

$|\varphi\rangle \rightarrow \varphi(r)$

$\langle\varphi| \rightarrow \varphi^*(r)$

$$\langle\varphi_1|\varphi_2\rangle = \int \varphi_1^* \varphi_2 \, dr.$$

$$\langle\varphi_1|\varphi_2\rangle^* = \langle\varphi_2|\varphi_1\rangle$$

from $\circledast$ $\quad c_n = \langle u_n|\psi\rangle.$

$$\sum |c_n|^2 = 1. \quad \text{normalization condition.}$$

$|\psi\rangle$ is normalized.

bra $\langle\psi|$ corresponding to ket $|\psi\rangle$

$$\langle\psi| = \sum_n \langle u_n| c_n^*$$

use orthonormality of basis $\quad c_n^* = \langle\psi|u_n\rangle$

$$= \langle u_n|\psi\rangle^*$$

Introduce the operator

$P_{|u_n\rangle} \equiv |u_n\rangle\langle u_n|$ acting on arbitrary vector $|V\rangle$.

$$P_{|u_n\rangle}|V\rangle = \underline{\underline{|u_n\rangle}}\langle u_n|V\rangle$$

gives a vector aligned along $|u_n\rangle$

<u>Moreover,</u> $\quad P^2_{|u_n\rangle} = (|u_n\rangle\langle u_n|)(|u_n\rangle\langle u_n|)$

$$= |u_n\rangle(\langle u_n|u_n\rangle)\langle u_n| = |u_n\rangle\langle u_n|$$

$$= P_{|u_n\rangle}$$

Thus. $P_{|u_n\rangle}$ is a projection operator onto basis vector $|u_n\rangle$

<u>Completeness implies</u>

$$\sum_n P_{|u_n\rangle} = P_{|u_1\rangle} + P_{|u_2\rangle} + \cdots$$

$$= \sum_n |n\rangle\langle n| = \underline{1}$$

Evolution of $|\psi\rangle$ in time $\Leftarrow$ Schrödinger Eqn.

$$i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle. \qquad \text{in general } H = H(\underline{t})$$

Hamiltonian — energy – an observable

$\Rightarrow H$ <u>hermitian</u>. $\quad H^\dagger = H$

We used the notation $O^{\dagger} = O^{t\,*}$, $O^{\dagger}_{nm} = O^{*}_{mn}$ | Hermitian conjugate of arbitrary operat.

O is Hermitian if

$$O = O^{\dagger} \qquad O_{nm} = O^{*}_{mn}$$

Consider an observable described by the Hermitian operator A. Matrix elements of A in $\{|u_n\rangle\}$

$$A_{nm} = \langle u_n | A | u_m \rangle = \langle u_n | \left(A | u_m \rangle \right)$$

$$A \text{ - Hermitian} \Rightarrow A_{nm} = A^{*}_{mn}$$

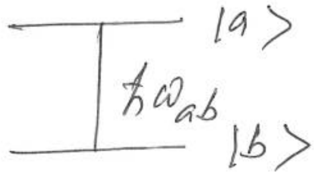
Mean value of A $\qquad \langle A \rangle = \langle \Psi | A | \Psi \rangle$

$$= \left(\sum_n c_n^{*} \langle u_n | \, A \sum_m c_m | u_m \rangle \right)$$

$$= \sum_{nm} c_n^{*} c_m \langle u_n | A | u_m \rangle$$

$$= \sum_{nm} c_n^{*} c_m A_{nm}$$

<u>Example</u>. <u>A two level system</u>.



$$E_a = \hbar\omega_a \qquad \langle a|a\rangle = \langle b|b\rangle = 1$$

$$E_b = \hbar\omega_b \qquad \langle a|b\rangle = \langle b|a\rangle = 0$$

<u>Completeness</u>. $\quad |a\rangle\langle a| + |b\rangle\langle b| = 1$.

$$|\Psi\rangle = C_a |a\rangle + C_b |b\rangle$$

$$|C_a|^2 + |C_b|^2 = 1$$

<u>C_a, C_b</u> prob. ampl.

bra $\langle\psi| = c_a^* \langle a| + c_b^* \langle b| = \langle a| c_a^* + \langle b| c_b^*$

mean value of A

$$\langle A \rangle = \left(\langle a| c_a^* + \langle b| c_b^* \right) A \left(c_a |a\rangle + c_b |b\rangle \right)$$

$$= |c_a|^2 \langle a|A|a\rangle + |c_b|^2 \langle b|A|b\rangle$$

$$+ c_a^* c_b \langle a|A|b\rangle + c_a c_b^* \langle b|A|a\rangle$$

$$= |c_a|^2 A_{aa} + |c_b|^2 A_{bb} + c_a^* c_b A_{ab} + c_b^* c_a A_{ba}$$

Total Hamiltonian. $H = H_0 + H_1$.

H_0 → free (unperturbed)

H_1 → interaction part.

$|a\rangle$ and $|b\rangle$ are the eigenstates of H_0

$$H_0 |j\rangle = \hbar \omega_j |j\rangle \quad j = a, b.$$

Since $H_0 = \hbar \omega_a |a\rangle\langle a| + \hbar \omega_b |b\rangle\langle b|$

We can write H_0

$$H_0 = \sum_{i,j} |i\rangle\langle i| H_0 |j\rangle\langle j|$$

$$= \sum |i\rangle\langle j| \left(\langle i|H_0|j\rangle \right)$$

$$= \sum_{ij} |i\rangle\langle j| \hbar \omega_j \delta_{ij} = \sum_j \hbar \omega_j |j\rangle\langle j|$$

Consider a particular case when the 2 level system is free ($H_1 = 0$)

$$i\hbar \frac{d}{dt} |\psi\rangle = H_0 |\psi\rangle$$

$$= \cancel{H_0} \left(\hbar \omega_a |a\rangle\langle a| + \hbar \omega_b |b\rangle\langle b| \right) |\psi\rangle$$

$$i\hbar \frac{d}{dt}\left(c_a|a\rangle + c_b|b\rangle\right) = \left(\hbar\omega_a|a\rangle\langle a| + \hbar\omega_b|b\rangle\langle b|\right)\left(c_a|a\rangle + c_b|b\rangle\right)$$

$$i\left(\dot{c}_a|a\rangle + \dot{c}_b|b\rangle\right) = \omega_a\, c_a|a\rangle + \omega_b\, c_b|b\rangle$$

$$\frac{d c_a}{dt} = -i\omega_a c_a. \qquad c_a(t) = e^{-i\omega_a t} c_a(0)$$

$$\frac{d c_b}{dt} = -i\omega_b c_b. \qquad c_b(t) = e^{-i\omega_b t} c_b(0)$$

$$\Longrightarrow |\psi(t)\rangle = c_a(t)|a\rangle + c_b(t)|b\rangle$$
$$= e^{-i\omega_a t} c_a(0)|a\rangle + e^{-i\omega_b t} c_b(0)|b\rangle$$

The probability of finding the free atom in the upper or lower levels are

$$|c_a(t)|^2 = |c_a(0)|^2 \text{ and } |c_b(t)|^2 = |c_b(0)|^2$$

$\Longrightarrow$ independent of time

In the contrary interference described by the cross terms

$$c_a(t)\, c_b^*(t) = e^{-i(\omega_a-\omega_b)t} c_a(0)\, c_b^*(0)$$

oscillates in time.

Ex. 1. σ is an operator such that

$\sigma|b\rangle = 0$, $\sigma|a\rangle = |b\rangle$ find σ.

$$\sigma|a\rangle\langle a| = |b\rangle\langle a| \qquad \sigma\left(|a\rangle\langle a| + |b\rangle\langle b|\right) = \underline{\underline{|b\rangle\langle a|}}$$

$$\sigma|b\rangle\langle b| = 0$$

Ex 2. by def $(O^\dagger)_{kl} = O^*_{lk}$

O is an arbitrary oper. and $O^\dagger$ its Hermit. conj.

Let $O = |u_n\rangle\langle u_m|$ show that $O^\dagger = |u_m\rangle\langle u_n|$.

$$O_{kl} = \langle u_k|u_n\rangle\langle u_m|u_l\rangle = \delta_{kn}\,\delta_{ml}.$$

Let $X = |u_m\rangle\langle u_n|$ $\quad X_{kl} = \delta_{km}\,\delta_{nl}$.

$$X_{kl} = O^*_{lk} = O^\dagger_{kl} \implies X = O^\dagger.$$

Ex 3. Let $O = |A\rangle\langle B|$ $\quad |A\rangle, |B\rangle$ two arbitrary vectors.

Action of O on $|\varphi\rangle$

$O|\varphi\rangle = |A\rangle(\langle B|\varphi\rangle)$ show that $\underline{O^\dagger = |B\rangle\langle A|}$

$$O_{kl} = \underbrace{\langle u_k|A\rangle}_{a_k}\underbrace{\langle B|u_l\rangle}_{b_l^*} = a_k\,b_l^*$$

Let $X = |B\rangle\langle A|$ $\quad X_{kl} = b_k\,a_l^* = O^*_{lk} = O^\dagger_{kl}$

$$\implies \underline{O^\dagger = |B\rangle\langle A|}.$$

Ex 4. $O = O_1 O_2$

$O^\dagger = O_2^\dagger O_1^\dagger$ | proof.

$$O^\dagger = (O_1 O_2)^\dagger = (O_1 O_2)^{t*} = (O_2^t O_1^t)^*$$

$$= O_2^{t*} O_1^{t*} = O_2^\dagger O_1^\dagger$$

Description by a density matrix

$$\langle A \rangle = \langle \psi | A | \psi \rangle = \sum_{nm} \underline{c_n^* c_m} A_{nm}$$

quadratic expressions.

$\Leftrightarrow$ matrix elements of the operator $|\psi\rangle\langle\psi|$

Projection onto ket $|\psi\rangle$. $\Leftarrow$

$$|\psi\rangle = \sum c_n |u_n\rangle$$

$$\langle\psi| = \sum_n c_m \langle u_m | c_m^*$$

$c_n = \langle u_n | \psi \rangle$

$c_n^* = \langle \psi | u_n \rangle$

$$\underline{c_n^* c_m = \langle \psi | u_n \rangle \langle u_m | \psi \rangle}$$

$$= \underline{\langle u_m | \psi \rangle \langle \psi | u_n \rangle}$$

natural to introduce the density operator $\Leftarrow$

$$\boxed{\rho = |\psi\rangle\langle\psi|.}$$

$$\rho_{mn} = \langle u_m | \rho | u_n \rangle$$

$$= \langle u_m | \psi \rangle \langle \psi | u_n \rangle$$

$$= c_m c_n^* = c_n^* c_m.$$

Mean value of an observable.

$$\langle A \rangle = \langle \psi | A | \psi \rangle$$

$$= \sum_{nm} c_n^* c_m A_{nm} = \sum_{mn} \rho_{mn} . A_{nm} = \mathrm{Tr}(\rho A)$$

$$= \mathrm{Tr}(A\rho)$$

Mathematically $|\psi\rangle\langle\psi|$ is a projection operator

$$P_{|\psi\rangle} = |\psi\rangle\langle\psi|$$

projects onto ket $|\psi\rangle$.

$$P_{|\psi\rangle} |V\rangle = |\psi\rangle (\langle \psi | V \rangle)$$

$$P_{|\psi\rangle}^2 = |\psi\rangle\langle\psi|\psi\rangle\langle\psi| = |\psi\rangle\langle\psi| = P_{|\psi\rangle} \Rightarrow P_{|\psi\rangle}^2 = P_{|\psi\rangle}$$

Projection operator.

In case of pure states $\rho^2 = \rho$

$$\mathrm{Tr}\,\rho^2 = \mathrm{Tr}\,\rho = 1. \quad \text{(see below)}$$

ρ_{mn} is Hermitian

$$\rho_{mn}^* = c_m^* c_n = \rho_{nm} \Longrightarrow \rho^\dagger = \rho.$$

Specification of ρ is enough to characterize a quantum state $\Rightarrow \rho$ enables us to obtain all the physical predictions that can be calculated from $|\psi\rangle$

Conservation of probability

$\underline{\mathrm{Tr}(\rho)}$: ⊛ $\sum |c_n|^2 = 1 = \sum_n \rho_{nn} = \mathrm{Tr}(\rho)$

$$\boxed{\mathrm{Tr}\,\rho = 1}.$$

⊛ $\langle A\rangle = \langle\psi|A|\psi\rangle = \mathrm{Tr}\{\rho A\} = \mathrm{Tr}\{A\rho\}$

⊛ Time evolution of ρ:

$$\frac{d}{dt}\langle\psi| = -\frac{1}{i\hbar}\langle\psi|H^\dagger = -\frac{1}{i\hbar}\langle\psi|H$$

$$\frac{d}{dt}|\psi\rangle = \frac{1}{i\hbar}H|\psi\rangle.$$

$$\frac{d}{dt}\rho = \frac{d}{dt}|\psi\rangle\langle\psi|$$

$$= \left(\frac{d|\psi\rangle}{dt}\right)\langle\psi| + |\psi\rangle\frac{d}{dt}\langle\psi|.$$

$$= \frac{1}{i\hbar}\left(H|\psi\rangle\langle\psi| \mp |\psi\rangle\langle\psi|H\right) = \frac{1}{i\hbar}(H\rho - \rho H)$$

$$\boxed{\frac{d\rho}{dt} = \frac{1}{i\hbar}[H,\rho]}$$ Generalized Schrödinger Eqn.

Properties of the density operator in case of pure states.

$\rho^\dagger = \rho$
$Tr\,\rho = 1$
$\langle A\rangle = Tr(\rho A) = Tr(A\rho)$
$i\hbar\frac{d\rho}{dt} = [H,\rho]$

These properties are general and holds also for mixed case. (mixed states)

In case of pure states: two specific properties

$$\circledast \quad \rho^2 = \rho, \quad Tr\,\rho^2 = 1$$

These can be used to find out if a state is pure or not.

In case of mixed states $\circledast$ no longer holds since ρ is no longer a projection operator.