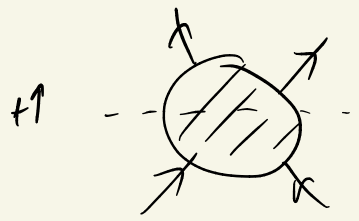
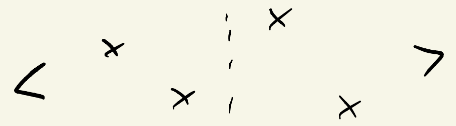
S-matrix and conformal bootstrap



$2\to 2$: E_{cm}, θ



Bootstrap: solve (or constrain) by self-consistency

Ingredients

i) symmetries Poincaré $\to E, J$ (J: $SO(d-1)$)

$M(E,\theta) = \sum_J a_J(E)\, P_J(\cos\theta)$ (P_J: spherical harmonics)

ii) positivity $|S_J| = |1 + i a_J| \le 1$

$0 \le \mathrm{Im}\, a_J \le 2$

(completeness relation: $S = 1 + iM$, $S^\dagger S = 1 \Rightarrow i(M^\dagger - M) = M^\dagger M \ge 0$)

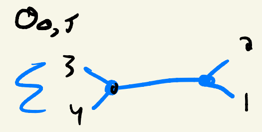
iii) crossing

conformal $\to \Delta, J$ (J: $SO(d)$)

$\langle \times\times\times\times \rangle = \sum_{\Delta,J} f^2_{\Delta J}\, G_{\Delta,J}(x\text{'s})$

$\boxed{f^2_{\Delta,J} \ge 0}$

$\langle \phi_4\phi_3 ; \phi_1\phi_2\rangle = \sum_{\mathcal{O}_{\Delta,J}} \langle\phi_4\phi_3|\mathcal{O}\rangle\langle\mathcal{O}|\phi_1\phi_2\rangle$

$= \sum$ 

$\langle 0|\phi_4\,[\phi_3, \phi_1]\,\phi_2|0\rangle = 0$ (1,3 spacelike)

$\Rightarrow \sum$  $= \sum$ 

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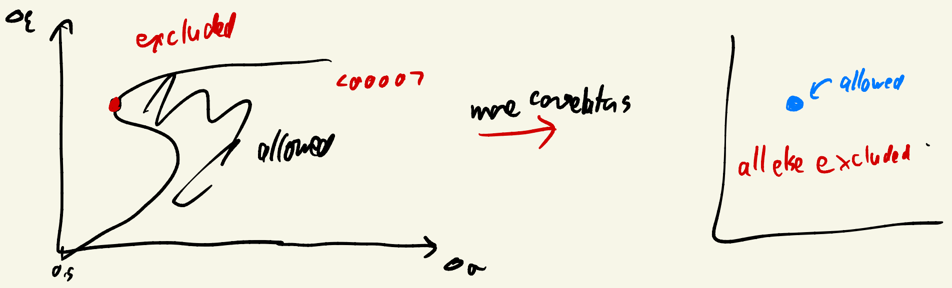
crossing $\Rightarrow \infty$ many constraints on ∞ many unknowns $= \begin{cases} \text{spectrum } \{\Delta, J\} \\ f^2_{\Delta,J} \text{ 3pt couplings} \end{cases}$

Ex: 3D critical Ising "Z_2-symmetry, only 2 relevant ops ($\Delta < 3$)"



Ex: pion S-matrices $\Rightarrow SU(2), \ldots$

Plan

Today: 0) outline

1) Unitarity + causality ($\to$ why)

2) $\hookrightarrow$ signal model analyticity

3) $\hookrightarrow$ crossing dispersion relations

Lecture 2: (S-matrix) EFT bootstrap

$\hookrightarrow$ manipulating inequalities

Lecture 3: EFT bootstrap with gravity, holographic CFTs

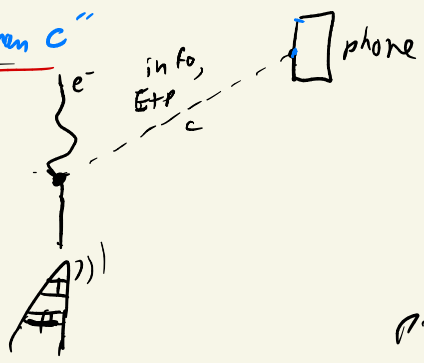
Refs: - "toolkit for analytic S-matrix" Correia, Sever + Zhiboedov

- CEMZ '14 (signal model)
Camanho, Edelstein, Maldacena + Zhiboedov (app D).

Unitarity: $|out\rangle = S|in\rangle \quad S^\dagger S = 1$

probabilities add up to 1 (and are positive)

Causality: "signals can't travel faster than c"



i) why waves

ii) why particles

iii) why antiparticles

...

why QFT

+1

less familiar: - why EFTs work
- why gravity attractive

In QFT: $[\ ,\ ] = 0$ spacelike

In Q grav: " " " in bdy (AdS / flat space)

bulk??

If one doesn't worry about "instantaneous action at a distance",
quantum gravity would be trivial:

$$H = \sum_i \frac{\vec{p}_i^{\,2}}{2m_i} + \sum_{i<j} V_{ij} \leftarrow \frac{-Gm_i m_j}{|\vec{x}_i - \vec{x}_j|}$$

↑ any function you want

↳ No predictive power!!!

Causality + unitarity: no real alternatives!

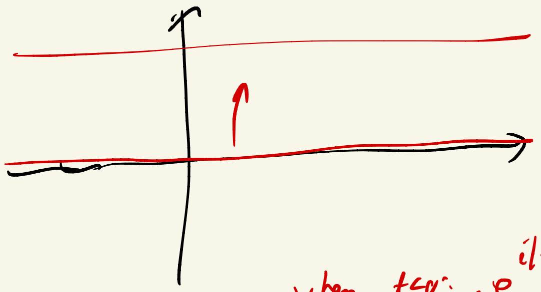
The puzzle is resolved by the following theorem: (Sommerfeld, Brillouin)

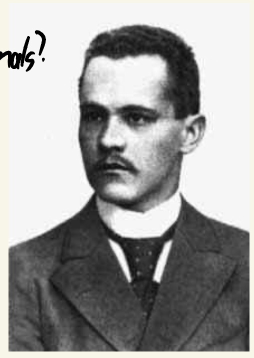
Let $F(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \hat{f}(\omega)$ ← analytic in UHP + sub-exp growth (*)

$\Rightarrow F(t) = 0,\ t<0.$ (Thm '1914)

proof: how to estimate oscillatory integrals?

trick: deform contour to make integrand as small as possible





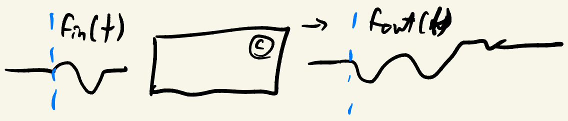
WKB

when $t<0$: $e^{i|t|\mathrm{Re}\,\omega - |t|\mathrm{Im}\,\omega}$ $f(\omega)$

decays as $\mathrm{Im}\,\omega \uparrow$; can't compete if sub-exp

for $f(\omega)$ non-analytic, or non sub-exp, we'll get stuck at some finite $\mathrm{Im}\,\omega$.

Signal model



Box is: - linear
- time-trans. invariant
- causal ($F_{out}(t) = 0$ for $t<0$ if so for $F_{in}(t)$)

$\hat{f}_{out}(\omega) = \boxed{S(\omega)}\, \hat{f}_{in}(\omega)$

claim: causality $\Rightarrow$ $S(\omega)$ is { analytic in UHP (*)
has sub-exponential growth at large $|\omega|$

naive argument: $F_{out}(t) = \int_{-\infty}^{t} dt'\, F_{in}(t')\, K(\overbrace{t-t'}^{\geq 0})$

$S(\omega) = \int_0^\infty dt\, K(t)\, e^{it\omega}$ $(\mathrm{Re}\,\omega + i\,\mathrm{Im})$

$\sim e^{-t\,\mathrm{Im}\,\omega}$

geometric optics limit:

$F_{in}(\omega)$ strongly peaked around ω_0 $\quad S(\omega) = e^{i\delta(\omega)}$ ↑ slowly varying

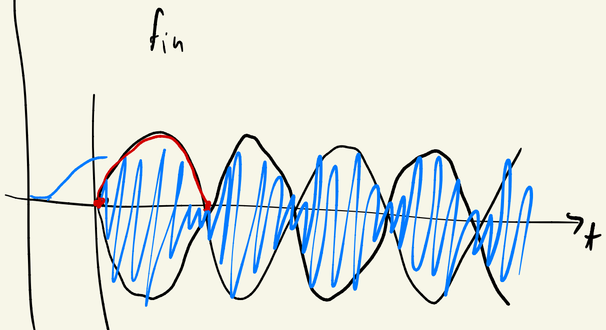
$$F_{out}(t) = \int \frac{d\omega}{2\pi} \hat{f}_{in}(\omega)\, e^{-i\omega t + i(\delta(\omega_0) + (\omega-\omega_0)\delta'(\omega_0) + \cdots)}$$

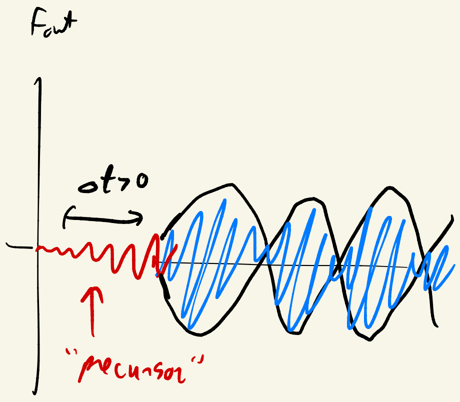
$$F_{out}(t) = e^{i\delta(\omega_0) - i\omega_0 t} F_{in}(t - \Delta t) \qquad \Delta t = \frac{d\delta}{d\omega}\Big|_{\omega_0}$$

similar concept as 'group velocity'

puzzle: $\frac{d\omega}{dt} < 0$ is possible!
(even if (*) is satisfied!) (problem 2)

Normal case ($\Delta t > 0$)





when $\Delta t < 0$ $\Rightarrow$ either large distortion or attenuation. +1

if Box unitary: $|S(\omega)| \leq 1$ on real axis

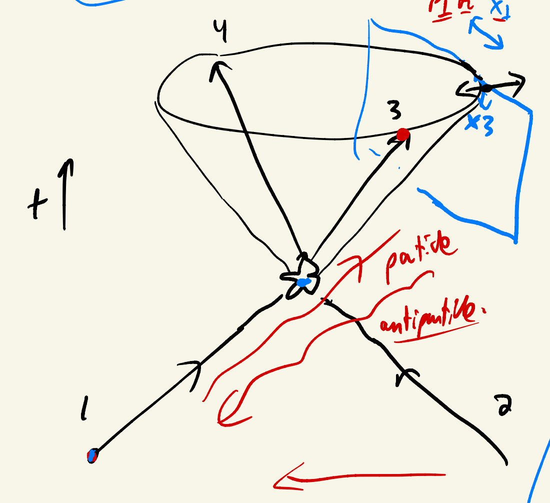
(<1 means particle got stuck) "inelastic" $\quad F_{in}(t) \Rightarrow \psi(x)$ (if $|\psi_{out}|^2 \leq |\psi_{in}|^2$)

Unitary + causal $\Rightarrow$ $\boxed{|S(\omega)| \leq 1 \text{ in full upper halfplane}}$

CEMZ '14

↳ bound on chaos (do problem 1!)

2→2 scattering



$\mathcal{M} = \langle 4 | \phi_3 \; \phi_1 | 2 \rangle$

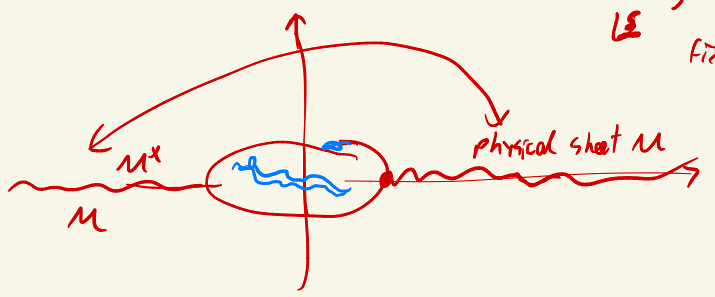
i) causality says nothing about finite angle scattering (LSZ)
(3 can't be spacelike from 1)
↳ Forward limits

ii) use $\langle 4 | [\phi_3, \phi_1] | 2 \rangle = 0$ spacelike
$\Rightarrow \langle 4 | 3 \; 1 | 2 \rangle(\theta) = \langle 4 | 1 \; 3 | 2 \rangle(4)$
$\mathcal{M}_{12\to 34}$ $\qquad$ $\mathcal{M}^*_{23\to 14}$

crossing symmetry

apply signal model.

⇒ why amplitude is analytic in UHP,



LS $s \sim E_1 E_2$
fixed t

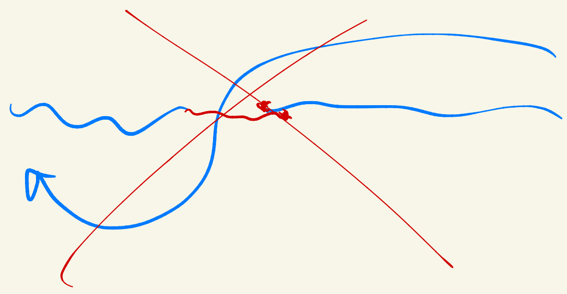
boundedness? $\quad \mathcal{M}(s,t)$ = analytic for $|s|$ large enough
but not bounded: plane waves ≠ normalizable

can't use $\mathcal{M}(b)$: would need $p_\perp^{200}$ ...!!

$\psi(p_\perp) \Rightarrow \mathcal{M}_\psi(s) = \frac{1}{2s}\int_0^{\mathcal{M}} dp_\perp \, \psi(p_\perp) \, \mathcal{M}(s, t = -p_\perp^2)$

normalizable in $b_\perp$ ⇒ bounded
and compact support in $p_\perp$ ⇒ analytic
(← regulate $\mathcal{E}_S$ sum)
apply signal model.
⇓

Axioms we'll use: $\quad \mathcal{M}_\psi(s) = \begin{cases} \text{analytic} \\ + \text{bounded} \end{cases}$ for: $|s| \geq s_0 \sim m^2$
$(|\mathcal{M}_\psi| \leq \text{const})$



$\mathcal{M}_s \xrightarrow{\text{fixed-}t} \mathcal{M}^*_u \xrightarrow{\text{fixed-}s} \mathcal{M}_t$